

# SELF-DUAL T-STRUCTURE

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**ABSTRACT.** We give a self-dual t-structure on the derived category of  $\mathbb{R}$ -constructible sheaves over any Noetherian regular ring by generalizing the notion of t-structure.

## INTRODUCTION

Let  $X$  be a complex manifold and let  $D_{\mathbb{C}-c}^b(\mathbf{k}_X)$  be the derived category of sheaves of  $\mathbf{k}$ -vector spaces on  $X$  with  $\mathbb{C}$ -constructible cohomologies. Here  $\mathbf{k}$  is a given base field. Then the t-structure  $({}^pD_{\mathbb{C}-c}^{\leq 0}(\mathbf{k}_X), {}^pD_{\mathbb{C}-c}^{\geq 0}(\mathbf{k}_X))$  on  $D_{\mathbb{C}-c}^b(\mathbf{k}_X)$  with middle perversity is self-dual with respect to the Verdier dual functor  $D_X = R\mathcal{H}om(\bullet, \omega_X)$ . Namely, the Verdier dual functor exchanges  ${}^pD_{\mathbb{C}-c}^{\leq 0}(\mathbf{k}_X)$  and  ${}^pD_{\mathbb{C}-c}^{\geq 0}(\mathbf{k}_X)$ . However, on a real analytic manifold  $X$  (of positive dimension), any perversity does not give a self-dual t-structure on the derived category  $D_{\mathbb{R}-c}^b(\mathbf{k}_X)$  of  $\mathbb{R}$ -constructible sheaves on  $X$ . In this paper, we construct such a self-dual t-structure after generalizing the notion of t-structure. This generalized notion already appeared in the paper of Bridgeland on stability conditions ([2]). (See also [4].) This construction can be also applied to the derived category  $D_{\text{coh}}^b(A)$  of finitely generated modules over a Noetherian regular ring  $A$ . We construct a (generalized) t-structure on  $D_{\text{coh}}^b(A)$  which is self-dual with respect to the duality functor  $R\text{Hom}_A(\bullet, A)$ .

Let us explain our results more precisely in the example of  $D_{\mathbb{R}-c}^b(\mathbf{k}_X)$ . Let  $X$  be a real analytic manifold. Recall that a sheaf  $F$  of  $\mathbf{k}$ -vector space is called  $\mathbb{R}$ -constructible if  $X$  is a locally finite union of locally closed subanalytic subsets  $\{X_\alpha\}_\alpha$  such that all the restrictions  $F|_{X_\alpha}$  are locally constant with finite-dimensional fibers. Let  $D_{\mathbb{R}-c}^b(\mathbf{k}_X)$  be the bounded derived category of  $\mathbb{R}$ -constructible sheaves. Let  $D_X = R\mathcal{H}om(\bullet, \omega_X)$  be the Verdier dual functor. For  $c \in \mathbb{R}$ , we define

$$(0.1) \quad \begin{aligned} {}^{1/2}D_{\mathbb{R}-c}^{\leq c}(\mathbf{k}_X) &:= \{K \in D_{\mathbb{R}-c}^b(\mathbf{k}_X) \mid \dim \text{Supp}(H^i K) \leq 2(c - i) \text{ for any } i \in \mathbb{Z}\}, \\ {}^{1/2}D_{\mathbb{R}-c}^{\geq c}(\mathbf{k}_X) &:= \{K \in D_{\mathbb{R}-c}^b(\mathbf{k}_X) \mid D_X K \in {}^{1/2}D_{\mathbb{R}-c}^{\leq -c}(\mathbf{k}_X)\}. \end{aligned}$$

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Then, the pair  $((^{1/2}D_{\mathbb{R}-c}^{\leq c}(\mathbf{k}_X))_{c \in \mathbb{R}}, (^{1/2}D_{\mathbb{R}-c}^{\geq c}(\mathbf{k}_X))_{c \in \mathbb{R}})$  satisfies the axiom of (generalized) t-structure (Definition 1.2). In particular,  $(^{1/2}D_{\mathbb{R}-c}^{\leq c}(\mathbf{k}_X), ^{1/2}D_{\mathbb{R}-c}^{> c-1}(\mathbf{k}_X))$  is a t-structure in the ordinary sense for any  $c \in \mathbb{R}$ . Here  $^{1/2}D_{\mathbb{R}-c}^{> c}(\mathbf{k}_X) := \bigcup_{b > c} ^{1/2}D_{\mathbb{R}-c}^{\geq b}(\mathbf{k}_X)$ . Therefore, for any  $K \in D_{\mathbb{R}-c}^b(\mathbf{k}_X)$  and  $c \in \mathbb{R}$ , there exists a distinguished triangle  $K' \rightarrow K \rightarrow K'' \xrightarrow{+1}$  in  $D_{\mathbb{R}-c}^b(\mathbf{k}_X)$  such that  $K' \in ^{1/2}D_{\mathbb{R}-c}^{\leq c}(\mathbf{k}_X)$  and  $K'' \in ^{1/2}D_{\mathbb{R}-c}^{> c}(\mathbf{k}_X)$ .

Note that  $^{1/2}D_{\mathbb{R}-c}^{\leq c}(\mathbf{k}_X) = ^{1/2}D_{\mathbb{R}-c}^{\leq s}(\mathbf{k}_X)$  for  $s \in \frac{1}{2}\mathbb{Z}$  such that  $s \leq c < s + 1/2$ , and  $^{1/2}D_{\mathbb{R}-c}^{> c}(\mathbf{k}_X) = ^{1/2}D_{\mathbb{R}-c}^{\geq s}(\mathbf{k}_X)$  for  $s \in \frac{1}{2}\mathbb{Z}$  such that  $s - 1/2 < c \leq s$ .

This paper is organized as follows. In Section 1, we generalize the notion of a t-structure. In Section 2, we recall the result of [4] on a t-structure on the derived category of a quasi-abelian category. In Section 3, we give the t-structure associated with a torsion pair on an abelian category.

In Section 4, we give a self-dual t-structure on the derived category of coherent sheaves on a Noetherian regular scheme.

In Section 5, we define two t-structures on the derived category of the abelian category of  $\mathbb{R}$ -constructible sheaves of  $A$ -modules on a subanalytic space  $X$ . Here  $A$  is a Noetherian regular ring. One is purely topological and the other is self-dual with respect to the Verdier duality functor.

In Section 6, we study the self-dual t-structure on the derived category of the abelian category of sheaves of  $A$ -modules on a complex manifold  $X$  with  $\mathbb{C}$ -constructible cohomologies. The main result is its microlocal characterization (Theorem 6.2).

**Convention.** In this paper, all subanalytic spaces and complex analytic spaces are assumed to be Hausdorff, locally compact, countable at infinity and with finite dimension.

## 1. (GENERALIZED) T-STRUCTURE

Since the following lemma is elementary, we omit its proof.

**Lemma 1.1.** *Let  $X$  be a set.*

- (i) *Let  $(X^{\leq c})_{c \in \mathbb{R}}$  be a family of subsets of  $X$  such that  $X^{\leq c} = \bigcap_{b > c} X^{\leq b}$  for any  $c \in \mathbb{R}$ . Set  $X^{< c} := \bigcup_{b < c} X^{\leq b}$ . Then we have*
  - (a)  $X^{< c} = \bigcup_{b < c} X^{< b}$ ,
  - (b)  $X^{\leq c} = \bigcap_{b > c} X^{< b}$ .
- (ii) *Conversely, let  $(X^{< c})_{c \in \mathbb{R}}$  be a family of subsets of  $X$  such that  $X^{< c} = \bigcup_{b < c} X^{< b}$  for any  $c \in \mathbb{R}$ . Set  $X^{\leq c} := \bigcap_{b > c} X^{< b}$ . Then we have*
  - (a)  $X^{\leq c} = \bigcap_{b > c} X^{\leq b}$ ,
  - (b)  $X^{< c} = \bigcup_{b < c} X^{\leq b}$ .
- (iii) *Let  $(X^{\leq c})_{c \in \mathbb{R}}$  and  $(X^{< c})_{c \in \mathbb{R}}$  be as above. Let  $a, b \in \mathbb{R}$  such that  $a < b$ . If  $X^{< c} = X^{\leq c}$  for any  $c$  such that  $a < c \leq b$ , then  $X^{\leq a} = X^{\leq b}$ .*

Let us recall the notion of t-structure (see [1]). Let  $\mathcal{T}$  be a triangulated category. Let  $\mathcal{T}^{\leq 0}$  and  $\mathcal{T}^{\geq 0}$  be strictly full subcategories of  $\mathcal{T}$ . Here, a subcategory  $\mathcal{C}'$  of a category  $\mathcal{C}$  is called *strictly full* if it is full, i.e.  $\text{Hom}_{\mathcal{C}'}(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y)$  for any  $X, Y \in \mathcal{C}'$ , and any object of  $\mathcal{C}$  isomorphic to some object of  $\mathcal{C}'$  is an object of  $\mathcal{C}'$ .

For  $n \in \mathbb{Z}$ , we set  $\mathcal{T}^{\leq n} = \mathcal{T}^{\leq 0}[-n]$  and  $\mathcal{T}^{\geq n} = \mathcal{T}^{\geq 0}[-n]$ . Let us recall that  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  is a t-structure on  $\mathcal{T}$  if it satisfies:

$$(1.1) \quad \begin{cases} (a) \mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 1} \text{ and } \mathcal{T}^{\geq 1} \subset \mathcal{T}^{\geq 0}, \\ (b) \text{Hom}_{\mathcal{T}}(X, Y) = 0 \text{ for } X \in \mathcal{T}^{\leq 0} \text{ and } Y \in \mathcal{T}^{\geq 1}, \\ (c) \text{ for any } X \in \mathcal{T}, \text{ there exists a distinguished triangle } X_0 \rightarrow X \rightarrow X_1 \xrightarrow{+1} \text{ in } \mathcal{T} \text{ such that } X_0 \in \mathcal{T}^{\leq 0} \text{ and } X_1 \in \mathcal{T}^{\geq 1}. \end{cases}$$

We shall generalize this notion.

**Definition 1.2.** Let  $(\mathcal{T}^{\leq c})_{c \in \mathbb{R}}$  and  $(\mathcal{T}^{\geq c})_{c \in \mathbb{R}}$  be families of strictly full subcategories of a triangulated category  $\mathcal{T}$ , and set  $\mathcal{T}^{< c} = \bigcup_{b < c} \mathcal{T}^{\leq b}$  and  $\mathcal{T}^{> c} = \bigcup_{b > c} \mathcal{T}^{\geq b}$ . We say that  $((\mathcal{T}^{\leq c})_{c \in \mathbb{R}}, (\mathcal{T}^{\geq c})_{c \in \mathbb{R}})$  is a (generalized) t-structure (cf. [2]) if it satisfies the following conditions.

$$(1.2) \quad \begin{cases} (a) \mathcal{T}^{\leq c} = \bigcap_{b > c} \mathcal{T}^{\leq b} \text{ and } \mathcal{T}^{\geq c} = \bigcap_{b < c} \mathcal{T}^{\geq b} \text{ for any } c \in \mathbb{R}, \\ (b) \mathcal{T}^{\leq c+1} = \mathcal{T}^{\leq c}[-1] \text{ and } \mathcal{T}^{\geq c+1} = \mathcal{T}^{\geq c}[-1] \text{ for any } c \in \mathbb{R}, \\ (c) \text{ we have } \text{Hom}_{\mathcal{T}}(X, Y) = 0 \text{ for any } c \in \mathbb{R}, X \in \mathcal{T}^{< c} \text{ and } Y \in \mathcal{T}^{> c}, \\ (d) \text{ for any } X \in \mathcal{T} \text{ and } c \in \mathbb{R}, \text{ there exist distinguished triangles } X_0 \rightarrow X \rightarrow X_1 \xrightarrow{+1} \text{ and } X'_0 \rightarrow X \rightarrow X'_1 \xrightarrow{+1} \text{ in } \mathcal{T} \text{ such that } X_0 \in \mathcal{T}^{\leq c}, X_1 \in \mathcal{T}^{> c} \text{ and } X'_0 \in \mathcal{T}^{< c}, X'_1 \in \mathcal{T}^{\geq c}. \end{cases}$$

Note that under conditions (a)–(c), the distinguished triangles in (d) are unique up to a unique isomorphism.

If  $((\mathcal{T}^{\leq c})_{c \in \mathbb{R}}, (\mathcal{T}^{\geq c})_{c \in \mathbb{R}})$  is a generalized t-structure, then the pairs  $(\mathcal{T}^{\leq c}, \mathcal{T}^{> c-1})$  and  $(\mathcal{T}^{< c}, \mathcal{T}^{\geq c-1})$  are t-structures in the original sense for any  $c \in \mathbb{R}$ . Hence,  $\mathcal{T}^{\leq c} \cap \mathcal{T}^{> c-1}$  and  $\mathcal{T}^{< c} \cap \mathcal{T}^{\geq c-1}$  are abelian categories.

Assume that  $((\mathcal{T}^{\leq c})_{c \in \mathbb{R}}, (\mathcal{T}^{\geq c})_{c \in \mathbb{R}})$  is a generalized t-structure. Then the inclusion functors  $\mathcal{T}^{\leq c} \rightarrow \mathcal{T}$  and  $\mathcal{T}^{< c} \rightarrow \mathcal{T}$  have right adjoints

$$\tau^{\leq c}: \mathcal{T} \rightarrow \mathcal{T}^{\leq c} \text{ and } \tau^{< c}: \mathcal{T} \rightarrow \mathcal{T}^{< c}, \text{ respectively.}$$

Similarly, the inclusion functors  $\mathcal{T}^{\geq c} \rightarrow \mathcal{T}$  and  $\mathcal{T}^{> c} \rightarrow \mathcal{T}$  have left adjoints

$$\tau^{\geq c}: \mathcal{T} \rightarrow \mathcal{T}^{\geq c} \text{ and } \tau^{> c}: \mathcal{T} \rightarrow \mathcal{T}^{> c}, \text{ respectively.}$$

We have distinguished triangles functorially in  $X \in \mathcal{T}$ :

$$\begin{aligned} \tau^{\leq c} X \rightarrow X \rightarrow \tau^{> c} X \xrightarrow{+1} \quad \text{and} \\ \tau^{< c} X \rightarrow X \rightarrow \tau^{\geq c} X \xrightarrow{+1}. \end{aligned}$$

These four functors are called the truncation functors of the generalized t-structure  $((\mathcal{T}^{\leq c})_{c \in \mathbb{R}}, (\mathcal{T}^{\geq c})_{c \in \mathbb{R}})$ .

For any  $a, b \in \mathbb{R}$ , we have isomorphisms of functors:

$$\begin{aligned} \tau^{\leq a} \circ \tau^{\leq b} &\simeq \tau^{\leq \min(a,b)}, & \tau^{\geq a} \circ \tau^{\geq b} &\simeq \tau^{\geq \max(a,b)}, \text{ and} \\ \tau^{\leq a} \circ \tau^{\geq b} &\simeq \tau^{\geq b} \circ \tau^{\leq a}. \end{aligned}$$

In the last formula, we can replace  $\tau^{\geq a}$  with  $\tau^{>a}$  or  $\tau^{\leq b}$  with  $\tau^{<b}$ . For any  $c \in \mathbb{R}$ , we have

$$\begin{aligned} (1.3) \quad \mathcal{T}^{\leq c} &= \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(X, Y) \simeq 0 \text{ for any } Y \in \mathcal{T}^{>c}\}, \\ \mathcal{T}^{<c} &= \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(X, Y) \simeq 0 \text{ for any } Y \in \mathcal{T}^{\geq c}\}, \\ \mathcal{T}^{\geq c} &= \{Y \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(X, Y) \simeq 0 \text{ for any } X \in \mathcal{T}^{<c}\}, \\ \mathcal{T}^{>c} &= \{Y \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(X, Y) \simeq 0 \text{ for any } X \in \mathcal{T}^{\leq c}\}. \end{aligned}$$

We set  $\mathcal{T}^c := \mathcal{T}^{\leq c} \cap \mathcal{T}^{\geq c}$ . Then  $\mathcal{T}^c$  is a quasi-abelian category (see [2] and [6]). More generally, for  $a \leq b$ , we set

$$\mathcal{T}^{[a,b]} := \mathcal{T}^{\leq b} \cap \mathcal{T}^{\geq a}.$$

Then  $\mathcal{T}^{[a,b]}$  is a quasi-abelian category if  $a \leq b < a + 1$ .

A t-structure  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  is regarded as a generalized t-structure by

$$\begin{aligned} (1.4) \quad \mathcal{T}^{\leq c} &= \mathcal{T}^{\leq 0}[-n] \quad \text{for } n \in \mathbb{Z} \text{ such that } n \leq c < n + 1, \\ \mathcal{T}^{\geq c} &= \mathcal{T}^{\geq 0}[-n] \quad \text{for } n \in \mathbb{Z} \text{ such that } n - 1 < c \leq n. \end{aligned}$$

Hence, a t-structure is nothing but a generalized t-structure such that  $\mathcal{T}^{\leq 0} = \mathcal{T}^{<1}$  and  $\mathcal{T}^{\geq 1} = \mathcal{T}^{>0}$ , or equivalently  $\mathcal{T}^c = 0$  for any  $c \notin \mathbb{Z}$ .

*In the sequel, we call a generalized t-structure simply a t-structure.*

**Remark 1.3.** In the examples we give in this paper, the t-structures also satisfy the following condition:

- (e) for any  $c \in \mathbb{R}$  we can find  $a$  and  $b$  such that  $a < c < b$  and
- (1)  $\mathcal{T}^{<c} = \mathcal{T}^{\leq a}$ ,  $\mathcal{T}^{\leq c} = \mathcal{T}^{<b}$ ,
  - (2)  $\mathcal{T}^{>c} = \mathcal{T}^{\geq b}$ ,  $\mathcal{T}^{\geq c} = \mathcal{T}^{>a}$ .

More precisely, in the examples in this paper, we can take  $a = \max\{s \in \frac{1}{2}\mathbb{Z} \mid s < c\}$  and  $b = \min\{s \in \frac{1}{2}\mathbb{Z} \mid s > c\}$ . Hence  $\mathcal{T}^c = 0$  if  $c \notin \frac{1}{2}\mathbb{Z}$ .

## 2. T-STRUCTURE ON THE DERIVED CATEGORY OF A QUASI-ABELIAN CATEGORY

For more details, see [4, § 2].

Let  $\mathcal{C}$  be a quasi-abelian category (see [6]). Recall that, for a morphism  $f: X \rightarrow Y$  in  $\mathcal{C}$ ,  $\text{Im}(f) := \text{Ker}(Y \rightarrow \text{Coker}(f))$  and  $\text{Coim}(f) := \text{Coker}(\text{Ker}(f) \rightarrow X)$ . Hence, we have a diagram

$$\text{Ker}(f) \longrightarrow X \xrightarrow{\quad} \text{Coim}(f) \longrightarrow \text{Im}(f) \longrightarrow Y \longrightarrow \text{Coker}(f) .$$

$f$

Let  $\text{C}(\mathcal{C})$  be the category of complexes in  $\mathcal{C}$ , and  $\text{D}(\mathcal{C})$  the derived category of  $\mathcal{C}$  (see [6]). Let us define various truncation functors for  $X \in \text{C}(\mathcal{C})$ :

$$\begin{aligned} \tau^{\leq n} X &: \dots \rightarrow X^{n-1} \rightarrow \text{Ker } d_X^n \rightarrow 0 \rightarrow 0 \rightarrow \dots \\ \tau^{\leq n+1/2} X &: \dots \rightarrow X^{n-1} \rightarrow X^n \rightarrow \text{Im } d_X^n \rightarrow 0 \rightarrow \dots \\ \tau^{\geq n} X &: \dots \rightarrow 0 \rightarrow \text{Coker } d_X^{n-1} \rightarrow X^{n+1} \rightarrow X^{n+2} \rightarrow \dots \\ \tau^{\geq n+1/2} X &: \dots \rightarrow 0 \rightarrow \text{Coim } d_X^n \rightarrow X^{n+1} \rightarrow X^{n+2} \rightarrow \dots \end{aligned}$$

for  $n \in \mathbb{Z}$ . Then we have morphisms functorial in  $X$ :

$$\tau^{\leq s} X \longrightarrow \tau^{\leq t} X \longrightarrow X \longrightarrow \tau^{\geq s} X \longrightarrow \tau^{\geq t} X$$

for  $s, t \in \frac{1}{2}\mathbb{Z}$  such that  $s \leq t$ . We can easily check that these functors  $\tau^{\leq s}, \tau^{\geq s}: \text{C}(\mathcal{C}) \rightarrow \text{C}(\mathcal{C})$  send the morphisms homotopic to zero to morphisms homotopic to zero and the quasi-isomorphisms to quasi-isomorphisms. Hence, they induce the functors

$$\tau^{\leq s}, \tau^{\geq s}: \text{D}(\mathcal{C}) \rightarrow \text{D}(\mathcal{C})$$

and morphisms  $\tau^{\leq s} \rightarrow \text{id} \rightarrow \tau^{\geq s}$ .

For  $s \in \frac{1}{2}\mathbb{Z}$ , set

$$\begin{aligned} \text{D}^{\leq s}(\mathcal{C}) &= \{X \in \text{D}(\mathcal{C}) \mid \tau^{\leq s} X \rightarrow X \text{ is an isomorphism}\} \\ \text{D}^{\geq s}(\mathcal{C}) &= \{X \in \text{D}(\mathcal{C}) \mid X \rightarrow \tau^{\geq s} X \text{ is an isomorphism}\} . \end{aligned}$$

Then  $\{\text{D}^{\leq s}(\mathcal{C})\}_{s \in \frac{1}{2}\mathbb{Z}}$  is an increasing sequence of strictly full subcategories of  $\text{D}(\mathcal{C})$ , and  $\{\text{D}^{\geq s}(\mathcal{C})\}_{s \in \frac{1}{2}\mathbb{Z}}$  is a decreasing sequence of strictly full subcategories of  $\text{D}(\mathcal{C})$ .

The functor  $\tau^{\leq s}: \text{D}(\mathcal{C}) \rightarrow \text{D}^{\leq s}(\mathcal{C})$  is a right adjoint functor of the inclusion functor  $\text{D}^{\leq s}(\mathcal{C}) \hookrightarrow \text{D}(\mathcal{C})$ , and  $\tau^{\geq s}: \text{D}(\mathcal{C}) \rightarrow \text{D}^{\geq s}(\mathcal{C})$  is a left adjoint functor of  $\text{D}^{\geq s}(\mathcal{C}) \hookrightarrow \text{D}(\mathcal{C})$ .

For  $c \in \mathbb{R}$ , we set

$$\begin{aligned} \text{D}^{\leq c}(\mathcal{C}) &= \text{D}^{\leq s}(\mathcal{C}) \quad \text{where } s \in \frac{1}{2}\mathbb{Z} \text{ satisfies } s \leq c < s + 1/2, \\ \text{D}^{\geq c}(\mathcal{C}) &= \text{D}^{\geq s}(\mathcal{C}) \quad \text{where } s \in \frac{1}{2}\mathbb{Z} \text{ satisfies } s - 1/2 < c \leq s. \end{aligned} \tag{2.1}$$

Then

**Proposition 2.1** ([6], see also [4]).  $((\text{D}^{\leq c}(\mathcal{C}))_{c \in \mathbb{R}}, (\text{D}^{\geq c}(\mathcal{C}))_{c \in \mathbb{R}})$  is a  $t$ -structure.

We call it the *standard t-structure* on  $D(\mathcal{C})$ . The triangulated category  $D(\mathcal{C})$  is equivalent to the derived category of the abelian category  $D^{\leq c}(\mathcal{C}) \cap D^{> c-1}(\mathcal{C})$  for every  $c \in \mathbb{R}$ . The full subcategory  $D^0(\mathcal{C}) := D^{\leq 0}(\mathcal{C}) \cap D^{\geq 0}(\mathcal{C})$  is equivalent to  $\mathcal{C}$ .

If  $\mathcal{C}$  is an abelian category, then the standard t-structure is:

$$\begin{aligned} D^{\leq c}(\mathcal{C}) &= \{X \in D(\mathcal{C}) \mid H^i(X) = 0 \text{ for any } i > c\}, \\ D^{\geq c}(\mathcal{C}) &= \{X \in D(\mathcal{C}) \mid H^i(X) = 0 \text{ for any } i < c\}. \end{aligned}$$

### 3. T-STRUCTURE ASSOCIATED WITH A TORSION PAIR

Let  $\mathcal{C}$  be an abelian category. A torsion pair is a pair  $(\mathsf{T}, \mathsf{F})$  of strictly full subcategories of  $\mathcal{C}$  such that

$$(3.1) \quad \begin{cases} \text{(a) } \operatorname{Hom}_{\mathcal{C}}(X, Y) = 0 \text{ for any } X \in \mathsf{T} \text{ and } Y \in \mathsf{F}, \\ \text{(b) for any } X \in \mathcal{C}, \text{ there exists an exact sequence } 0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0 \\ \text{with } X' \in \mathsf{T} \text{ and } X'' \in \mathsf{F}. \end{cases}$$

Let  $(\mathsf{T}, \mathsf{F})$  be a torsion pair. Then we have

$$\begin{aligned} \mathsf{T} &\simeq \{X \in \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(X, Y) = 0 \text{ for any } Y \in \mathsf{F}\}, \\ \mathsf{F} &\simeq \{Y \in \mathcal{C} \mid \operatorname{Hom}_{\mathcal{C}}(X, Y) = 0 \text{ for any } X \in \mathsf{T}\}. \end{aligned}$$

Moreover,  $\mathsf{T}$  is stable under taking quotients and extensions, while  $\mathsf{F}$  is stable under taking subobjects and extensions.

For any integer  $n$ , we define

$$\begin{aligned} {}^pD^{\leq n}(\mathcal{C}) &:= \{X \in D(\mathcal{C}) \mid H^i(X) \simeq 0 \text{ for any } i > n\}, \\ {}^pD^{\leq n-1/2}(\mathcal{C}) &:= \{X \in D(\mathcal{C}) \mid H^i(X) \simeq 0 \text{ for any } i > n \text{ and } H^n(X) \in \mathsf{T}\}, \\ {}^pD^{\geq n-1/2}(\mathcal{C}) &:= \{X \in D(\mathcal{C}) \mid H^i(X) \simeq 0 \text{ for any } i < n\}, \\ {}^pD^{\geq n}(\mathcal{C}) &:= \{X \in D(\mathcal{C})^{\geq n-1/2} \mid H^i(X) \simeq 0 \text{ for any } i < n \text{ and } H^n(X) \in \mathsf{F}\}. \end{aligned} \quad (3.2)$$

For any  $c \in \mathbb{R}$ , we define  ${}^pD^{\leq c}(\mathcal{C})$  and  ${}^pD^{\geq c}(\mathcal{C})$  by (2.1).

Since the following proposition can be easily proved, we omit the proof.

**Proposition 3.1.**  $(({}^pD^{\leq c}(\mathcal{C}))_{c \in \mathbb{R}}, ({}^pD^{\geq c}(\mathcal{C}))_{c \in \mathbb{R}})$  is a t-structure.

We have

$$\mathsf{T} \simeq {}^pD^{-1/2}(\mathcal{C}), \quad \mathsf{F} \simeq {}^pD^0(\mathcal{C}), \quad \text{and} \quad \mathcal{C} \simeq {}^pD^{[-1/2, 0]}(\mathcal{C}).$$

Moreover,  $D(\mathcal{C})$  is equivalent to the derived category of the abelian category  ${}^pD^{[0, 1/2]}(\mathcal{C})$ .

Note that we have

$$D^{\leq c}(\mathcal{C}) \subset {}^pD^{\leq c}(\mathcal{C}) \subset D^{\leq c+1/2}(\mathcal{C}) \quad \text{and} \quad D^{\geq c+1/2}(\mathcal{C}) \subset {}^pD^{\geq c}(\mathcal{C}) \subset D^{\geq c}(\mathcal{C}).$$

## 4. SELF-DUAL T-STRUCTURE ON THE DERIVED CATEGORY OF COHERENT SHEAVES

Let  $X$  be a Noetherian regular scheme. Let  $D_X$  be the duality functor  $D_X := R\mathcal{H}om_{\mathcal{O}_X}(\bullet, \mathcal{O}_X)$ . Let  $D_{\text{coh}}^b(\mathcal{O}_X)$  be the bounded derived category of  $\mathcal{O}_X$ -modules with coherent cohomologies. We denote by  $((D_{\text{coh}}^{\leq c}(\mathcal{O}_X))_{c \in \mathbb{R}}, (D_{\text{coh}}^{\geq c}(\mathcal{O}_X))_{c \in \mathbb{R}})$  the standard t-structure on  $D_{\text{coh}}^b(\mathcal{O}_X)$ .

Recall that, for any coherent  $\mathcal{O}_X$ -module  $\mathcal{F}$ , its codimension is defined by

$$\text{codim}(\mathcal{F}) := \text{codim}(\text{Supp}(\mathcal{F})) = \inf_{x \in \text{Supp}(\mathcal{F})} \dim \mathcal{O}_{X,x}.$$

Here we understand  $\text{codim}(0) = +\infty$ .

We set

$$\begin{aligned} {}^{1/2}D_{\text{coh}}^{\leq c}(\mathcal{O}_X) &:= \{ \mathcal{F} \in D_{\text{coh}}^b(\mathcal{O}_X) \mid \text{codim}(H^i(\mathcal{F})) \geq 2(i - c) \text{ for any } i \in \mathbb{Z} \}, \\ {}^{1/2}D_{\text{coh}}^{\geq c}(\mathcal{O}_X) &:= \{ \mathcal{F} \in D_{\text{coh}}^b(\mathcal{O}_X) \mid D_X \mathcal{F} \in D_{\text{coh}}^{\leq -c}(\mathcal{O}_X) \} \\ &= \{ \mathcal{F} \in D_{\text{coh}}^b(\mathcal{O}_X) \mid \text{codim}(H^i(D_X \mathcal{F})) \geq 2(i + c) \text{ for any } i \in \mathbb{Z} \}. \end{aligned}$$

Then they satisfy Definition 1.2 (a). Remark that we have

$${}^{1/2}D_{\text{coh}}^{\leq c}(\mathcal{O}_X) = \left\{ \mathcal{F} \in D_{\text{coh}}^b(\mathcal{O}_X) \mid \mathcal{F}_x \in D^{\leq c + \frac{1}{2} \dim \mathcal{O}_{X,x}}(\mathcal{O}_{X,x}) \text{ for any } x \in X \right\}.$$

We have also

$$\begin{aligned} {}^{1/2}D_{\text{coh}}^{< c}(\mathcal{O}_X) &:= \bigcup_{b < c} {}^{1/2}D_{\text{coh}}^{\leq b}(\mathcal{O}_X) \\ &= \{ \mathcal{F} \in D_{\text{coh}}^b(\mathcal{O}_X) \mid \text{codim}(H^i(\mathcal{F})) > 2(i - c) \text{ for any } i \in \mathbb{Z} \}, \\ {}^{1/2}D_{\text{coh}}^{> c}(\mathcal{O}_X) &:= \bigcup_{b > c} {}^{1/2}D_{\text{coh}}^{\geq b}(\mathcal{O}_X) \\ &= \{ \mathcal{F} \in D_{\text{coh}}^b(\mathcal{O}_X) \mid \text{codim}(H^i(D_X \mathcal{F})) > 2(i + c) \text{ for any } i \in \mathbb{Z} \}. \end{aligned}$$

**Lemma 4.1.** *Let  $\mathcal{F} \in D_{\text{coh}}^b(\mathcal{O}_X)$ . Then  $\mathcal{F} \in {}^{1/2}D_{\text{coh}}^{\geq c}(\mathcal{O}_X)$  if and only if we have  $H^i R\Gamma_Z \mathcal{F} = 0$  for any closed subset  $Z$  and  $i < c + \text{codim} Z / 2$ .*

*Proof.* We shall use the results in [3]. Let us define the systems of support

$$\begin{aligned} \Phi^n &= \{ Z \mid \text{codim} Z \geq 2(n + c) \}, \\ \Psi^n &= \{ Z \mid n < c + 1 + \text{codim} Z / 2 \}. \end{aligned}$$

Then it is enough to show that

$$(4.1) \quad (\Phi \circ \Psi)^n := \bigcup_{i+j=n} (\Phi^i \cap \Psi^j) = \{ Z \mid \text{codim} Z \geq n \}.$$

Indeed, one has

$${}^{1/2}D_{\text{coh}}^{\leq -c}(\mathcal{O}_X) = {}^\Phi D_{\text{coh}}^{\leq 0}(\mathcal{O}_X) := \{ \mathcal{F} \in D_{\text{coh}}^b(\mathcal{O}_X) \mid \text{Supp}(H^k(\mathcal{F})) \in \Phi^k \text{ for any } k \in \mathbb{Z} \}$$

and hence [3, Theorem 5.9] along with (4.1) implies that  ${}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c}(\mathcal{O}_X)$  coincides with

$$\begin{aligned} {}^{\Psi}\mathrm{D}_{\mathrm{coh}}^{\geq 0}(\mathcal{O}_X) &:= \{F \mid H^i(\mathrm{R}\Gamma_Z F) = 0 \text{ for any } Z \in \Psi^{i+1}\} \\ &= \{F \mid H^i(\mathrm{R}\Gamma_Z F) = 0 \text{ for any } i < c + \mathrm{codim} Z/2\}. \end{aligned}$$

Let us show (4.1) Assume that  $Z \in \Phi^i \cap \Psi^j$  with  $i + j = n$ . Then we have

$$2 \mathrm{codim} Z \geq 2(i + c) + (2(j - c - 1) + 1) = 2n - 1$$

and hence  $\mathrm{codim} Z \geq n$ .

Conversely assume that  $\mathrm{codim} Z \geq n$ . Then take an integer  $i$  such that  $i \leq \mathrm{codim} Z/2 - c < i + 1$ . Then we have  $i > \mathrm{codim} Z/2 - c - 1$  and

$$j := n - i < \mathrm{codim} Z - (\mathrm{codim} Z/2 - c - 1) = c + 1 + \mathrm{codim} Z/2.$$

Hence  $Z \in \Phi^i \cap \Psi^j \subset (\Phi \circ \Psi)^n$ .  $\square$

**Proposition 4.2.**  $(({}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\leq c}(\mathcal{O}_X))_{c \in \mathbb{R}}, ({}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c}(\mathcal{O}_X))_{c \in \mathbb{R}})$  is a  $t$ -structure on  $\mathrm{D}_{\mathrm{coh}}^b(\mathcal{O}_X)$ .

*Proof.* It follows from [3]. Indeed, the pair  $({}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\leq c+1}(\mathcal{O}_X), {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c}(\mathcal{O}_X))$  coincides with  $({}^{\Psi}\mathrm{D}_{\mathrm{coh}}^b(\mathcal{O}_X)^{\leq 0}, {}^{\Psi}\mathrm{D}_{\mathrm{coh}}^b(\mathcal{O}_X)^{\geq 0})$  by the proof of the preceding proposition.  $\square$

**Corollary 4.3.** For  $\mathcal{F} \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\leq c}(\mathcal{O}_X)$  and  $\mathcal{G} \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c'}(\mathcal{O}_X)$ , we have

$$\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathrm{D}_{\mathrm{coh}}^{\geq c' - c}(\mathcal{O}_X).$$

Conversely we have

$${}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c'}(\mathcal{O}_X) = \{\mathcal{G} \in \mathrm{D}_{\mathrm{coh}}^b(\mathcal{O}_X) \mid \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathrm{D}_{\mathrm{coh}}^{\geq c' - c}(\mathcal{O}_X) \text{ for any } c \in \mathbb{R} \text{ and } \mathcal{F} \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\leq c}(\mathcal{O}_X)\} \quad \text{for any } c' \in \mathbb{R},$$

$${}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c}(\mathcal{O}_X) = \{\mathcal{F} \in \mathrm{D}_{\mathrm{coh}}^b(\mathcal{O}_X) \mid \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in \mathrm{D}_{\mathrm{coh}}^{\geq c' - c}(\mathcal{O}_X) \text{ for any } c' \in \mathbb{R} \text{ and } \mathcal{G} \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c'}(\mathcal{O}_X)\} \quad \text{for any } c \in \mathbb{R}.$$

**Proposition 4.4.** For  $\mathcal{F}, \mathcal{G} \in \mathrm{D}_{\mathrm{coh}}^b(\mathcal{O}_X)$ , we have

- (i) if  $\mathcal{F} \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\leq c}(\mathcal{O}_X)$  and  $\mathcal{G} \in \mathrm{D}_{\mathrm{coh}}^{\leq c'}(\mathcal{O}_X)$ , then  $\mathcal{F} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathcal{G} \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\leq c+c'}(\mathcal{O}_X)$ ,
- (ii) if  $\mathcal{F} \in \mathrm{D}_{\mathrm{coh}}^{\leq c}(\mathcal{O}_X)$  and  $\mathcal{G} \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c'}(\mathcal{O}_X)$ , then  $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c' - c}(\mathcal{O}_X)$ ,
- (iii) if  $\mathcal{F} \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c}(\mathcal{O}_X)$  and  $\mathcal{G} \in \mathrm{D}_{\mathrm{coh}}^{\leq c'}(\mathcal{O}_X)$ , then  $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\leq c' - c}(\mathcal{O}_X)$ ,
- (iv) if  $\mathcal{F} \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c}(\mathcal{O}_X)$  and  $\mathcal{G} \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c'}(\mathcal{O}_X)$ , then  $\mathcal{F} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathcal{G} \in \mathrm{D}_{\mathrm{coh}}^{\geq c+c'}(\mathcal{O}_X)$ .

*Proof.* (i) For any  $\mathcal{H} \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c''}(\mathcal{O}_X)$ , we have  $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}) \in \mathrm{D}_{\mathrm{coh}}^{\geq c'' - c}(\mathcal{O}_X)$  by Corollary 4.3. Hence,  $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathcal{G}, \mathcal{H}) \simeq \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{H}))$  belongs to  $\mathrm{D}_{\mathrm{coh}}^{\geq c'' - c - c'}(\mathcal{O}_X)$ . Since it holds for an arbitrary  $\mathcal{H} \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\geq c''}(\mathcal{O}_X)$ , we conclude  $\mathcal{F} \overset{\mathrm{L}}{\otimes}_{\mathcal{O}_X} \mathcal{G} \in {}^{1/2}\mathrm{D}_{\mathrm{coh}}^{\leq c+c'}(\mathcal{O}_X)$  by (1.3).



(ii) Since  $\mathcal{F} \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathbf{D}_X \mathcal{G} \in {}^{1/2}\mathbf{D}_{\text{coh}}^{\leq c-c'}(\mathcal{O}_X)$  by (i), and hence  $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq \mathbf{D}_X(\mathcal{F} \otimes^{\mathbf{L}} \mathbf{D}_X \mathcal{G})$  belongs to  ${}^{1/2}\mathbf{D}_{\text{coh}}^{\geq c'-c}(\mathcal{O}_X)$ .

(iii) Since  $\mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \simeq (\mathbf{D}_X \mathcal{F}) \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G}$ , (iii) follows from (i).

(iv) follows from Corollary 4.3 and  $\mathcal{F} \otimes_{\mathcal{O}_X}^{\mathbf{L}} \mathcal{G} \simeq \mathbf{R}\mathcal{H}om_{\mathcal{O}_X}(\mathbf{D}_X \mathcal{F}, \mathcal{G})$ .  $\square$

Let  $A$  be a Noetherian regular ring and  $X = \text{Spec}(A)$ . We write  $\mathbf{D}_{\text{coh}}^{\mathbf{b}}(A)$ ,  ${}^{1/2}\mathbf{D}_{\text{coh}}^{\leq c}(A)$  and  ${}^{1/2}\mathbf{D}_{\text{coh}}^{\geq c}(A)$  for  $\mathbf{D}_{\text{coh}}^{\mathbf{b}}(\mathcal{O}_X)$ ,  ${}^{1/2}\mathbf{D}_{\text{coh}}^{\leq c}(\mathcal{O}_X)$  and  ${}^{1/2}\mathbf{D}_{\text{coh}}^{\geq c}(\mathcal{O}_X)$ , respectively.

**Remark 4.5.** (i) A similar construction is possible for a complex manifold  $X$  and coherent  $\mathcal{O}_X$ -modules.

(ii) For any  $c \in \mathbb{R}$ , we have

$$\begin{aligned} \mathbf{D}_{\text{coh}}^{\leq c}(\mathcal{O}_X) &\subset {}^{1/2}\mathbf{D}_{\text{coh}}^{\leq c}(\mathcal{O}_X) \subset \mathbf{D}_{\text{coh}}^{\leq c+\dim X/2}(\mathcal{O}_X) \quad \text{and} \\ \mathbf{D}_{\text{coh}}^{\geq c+\dim X/2}(\mathcal{O}_X) &\subset {}^{1/2}\mathbf{D}_{\text{coh}}^{\geq c}(\mathcal{O}_X) \subset \mathbf{D}_{\text{coh}}^{\geq c}(\mathcal{O}_X). \end{aligned}$$

(iii) If  $\mathcal{F}$  is a Cohen-Macaulay  $\mathcal{O}_X$ -module with  $\text{codim}(\mathcal{F}) = r$ , then we have  $\mathcal{F} \in {}^{1/2}\mathbf{D}_{\text{coh}}^{-r/2}(\mathcal{O}_X)$ .

(iv) Assume that  $A$  is a Noetherian regular integral domain of dimension 1, and  $K$  the fraction field of  $A$ . Let  $\mathcal{C} = \text{Mod}_{\text{coh}}(A)$ . We take as  $\mathbf{T} \subset \mathcal{C}$  the subcategory of torsion  $A$ -modules, and as  $\mathbf{F}$  the subcategory of torsion free  $A$ -modules. Then the t-structure  $(({}^{\mathbf{p}}\mathbf{D}^{\leq c}(\mathcal{C}))_{c \in \mathbb{R}}, ({}^{\mathbf{p}}\mathbf{D}^{\geq c}(\mathcal{C}))_{c \in \mathbb{R}})$  associated with the torsion pair  $(\mathbf{T}, \mathbf{F})$  (see § 3) coincides with the t-structure  $(({}^{1/2}\mathbf{D}_{\text{coh}}^{\leq c}(A))_{c \in \mathbb{R}}, ({}^{1/2}\mathbf{D}_{\text{coh}}^{\geq c}(A))_{c \in \mathbb{R}})$ . Hence we have

$$(4.2) \quad \left\{ \begin{array}{l} {}^{1/2}\mathbf{D}_{\text{coh}}^{\leq n}(A) = \mathbf{D}_{\text{coh}}^{\leq n}(A), \\ {}^{1/2}\mathbf{D}_{\text{coh}}^{\leq n-1/2}(A) = \{X \in \mathbf{D}_{\text{coh}}^{\leq n}(A) \mid K \otimes_A X \in \mathbf{D}^{\leq n-1}(K)\}, \\ {}^{1/2}\mathbf{D}_{\text{coh}}^{\geq n-1/2}(A) = \mathbf{D}_{\text{coh}}^{\geq n}(A), \\ {}^{1/2}\mathbf{D}_{\text{coh}}^{\geq n}(A) = \{X \in \mathbf{D}_{\text{coh}}^{\geq n}(A) \mid H^n(X) \text{ is torsion free}\}. \end{array} \right.$$

for any  $n \in \mathbb{Z}$ .

Let  $\mathcal{F}$  be the quasi-abelian category of finitely generated torsion free  $A$ -modules. Then  $\mathbf{D}^{\mathbf{b}}(\mathcal{F}) \simeq \mathbf{D}_{\text{coh}}^{\mathbf{b}}(A)$ , and the t-structure  $(({}^{1/2}\mathbf{D}_{\text{coh}}^{\leq c}(A))_{c \in \mathbb{R}}, ({}^{1/2}\mathbf{D}_{\text{coh}}^{\geq c}(A))_{c \in \mathbb{R}})$  coincides with the standard t-structure of  $\mathbf{D}^{\mathbf{b}}(\mathcal{F})$ .

## 5. SELF-DUAL T-STRUCTURE: REAL CASE

**5.1. Topological perversity.** Let  $X$  be a subanalytic space (cf. [5, Exercise IX.2]). A subanalytic space is called smooth if it is locally isomorphic to a real analytic manifold as a subanalytic space.

A subanalytic stratification  $X = \bigsqcup_{\alpha \in I} X_\alpha$  of  $X$  is a locally finite family of locally closed smooth subanalytic subsets  $\{X_\alpha\}_{\alpha \in I}$  (called strata) such that the closure  $\overline{X_\alpha}$  is a union of strata for any  $\alpha$ . A subanalytic stratification  $X = \bigsqcup_{\alpha \in I} X_\alpha$  is called *good* if it satisfies the following condition:

- (5.1) for any  $K \in \mathrm{D}^b(\mathbb{Z}_X)$  such that  $K|_{X_\alpha}$  has locally constant cohomologies for all  $\alpha$ ,  $(\mathrm{R}\Gamma_{X_\alpha} K)|_{X_\alpha}$  has locally constant cohomologies for all  $\alpha$ .

Let  $X = \bigsqcup_{\alpha \in I} X_\alpha$  and  $X = \bigsqcup_{\alpha \in I'} X'_\alpha$  be two stratifications. We say that  $X = \bigsqcup_{\alpha \in I} X_\alpha$  is finer than  $X = \bigsqcup_{\alpha \in I'} X'_\alpha$  if any  $X_\alpha$  is contained in some  $X'_\beta$ . The following fact guarantees that there exist enough good stratifications:

- (5.2) For any locally finite family  $\{Z_j\}_j$  of locally closed subsets, there exists a good stratification such that any  $Z_j$  is a union of strata.

A regular subanalytic filtration of  $X$  is an increasing sequence

$$\emptyset = X_{-1} \subset \cdots \subset X_N = X$$

of closed subanalytic subsets  $X_k$  of  $X$  such that  $\overset{\circ}{X}_k := X_k \setminus X_{k-1}$  is smooth of dimension  $k$ . We say that it is a good filtration if  $\{\overset{\circ}{X}_k\}$  satisfies (5.1). Note that any subanalytic stratification  $X = \bigsqcup_{\alpha \in I} X_\alpha$  gives a regular subanalytic filtration defined by  $X_k := \bigsqcup_{\dim X_\alpha \leq k} X_\alpha$ .

Let  $A$  be a Noetherian regular ring. Let us denote by  $\mathrm{Mod}_{\mathbb{R}\text{-c}}(A_X)$  the category of  $\mathbb{R}$ -constructible  $A_X$ -modules, and by  $\mathrm{D}_{\mathbb{R}\text{-c}}^b(A_X)$  the bounded derived category of  $\mathbb{R}$ -constructible  $A_X$ -modules. We denote by  $((\mathrm{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_X))_{c \in \mathbb{R}}, (\mathrm{D}_{\mathbb{R}\text{-c}}^{\geq c}(A_X))_{c \in \mathbb{R}})$  the standard t-structure of  $\mathrm{D}_{\mathbb{R}\text{-c}}^b(A_X)$ , that is

$$\begin{aligned} \mathrm{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_X) &= \{K \in \mathrm{D}_{\mathbb{R}\text{-c}}^b(A_X) \mid H^i(K) = 0 \text{ for any } i > c\}, \\ \mathrm{D}_{\mathbb{R}\text{-c}}^{\geq c}(A_X) &= \{K \in \mathrm{D}_{\mathbb{R}\text{-c}}^b(A_X) \mid H^i(K) = 0 \text{ for any } i < c\}. \end{aligned}$$

We define

$$(5.3) \quad \begin{cases} {}^{1/2}_{\mathrm{KS}} \mathrm{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_X) = \{K \in \mathrm{D}_{\mathbb{R}\text{-c}}^b(A_X) \mid \dim \mathrm{Supp}(H^i(K)) \leq -2(i-c) \text{ for any } i\}, \\ {}^{1/2}_{\mathrm{KS}} \mathrm{D}_{\mathbb{R}\text{-c}}^{\geq c}(A_X) = \{K \in \mathrm{D}_{\mathbb{R}\text{-c}}^b(A_X) \mid H^i \mathrm{R}\Gamma_Z(K) = 0 \text{ for any closed subanalytic subset } Z \text{ and } i < c - \dim Z/2\}. \end{cases}$$

Then we have

**Proposition 5.1.** *The pair  $((^{1/2}_{\text{KS}} D_{\mathbb{R}-c}^{\leq c}(A_X))_{c \in \mathbb{R}}, (^{1/2}_{\text{KS}} D_{\mathbb{R}-c}^{\geq c}(A_X))_{c \in \mathbb{R}})$  is a t-structure on  $D_{\mathbb{R}-c}^b(A_X)$ .*

*Proof.* Indeed,  $(^{1/2}_{\text{KS}} D_{\mathbb{R}-c}^{\leq c+1}(A_X), ^{1/2}_{\text{KS}} D_{\mathbb{R}-c}^{\geq c}(A_X))$  coincides with the t-structure associated with the perversity  $p(n) = \lceil c - n/2 \rceil$  (see e.g. [5, Definition 10.2.1]).  $\square$

**Lemma 5.2** ([5, Proposition 10.2.4]). *Let  $K \in D_{\mathbb{R}-c}^b(A_X)$  and let  $X = \bigsqcup_{\alpha} X_{\alpha}$  be a subanalytic stratification of  $X$  such that  $(D_X K)|_{X_{\alpha}}$  has locally constant cohomologies for any  $\alpha$ . Then  $K \in ^{1/2}_{\text{KS}} D_{\mathbb{R}-c}^{\geq c}(A_X)$  if and only if  $(R\Gamma_{X_{\alpha}} K)_x \in D_{\text{coh}}^{\geq c - \dim X_{\alpha}/2}(A)$  for any  $\alpha$  and  $x \in X_{\alpha}$ .*

**5.2. Self-dual t-structure :  $\mathbb{R}$ -constructible case.** As in the preceding subsection,  $X$  is a subanalytic space and  $A$  is a Noetherian regular ring. Let  $D_X$  be the duality functor

$$D_X(K) = R\mathcal{H}om_A(K, \omega_X) \quad \text{for } K \in D_{\mathbb{R}-c}^b(A_X),$$

where  $\omega_X = a_X^! A_{\text{pt}}$  with the canonical projection  $a_X: X \rightarrow \text{pt}$ .

For  $F \in \text{Mod}_{\mathbb{R}-c}(A_X)$ , we set

$$(5.4) \quad \text{mod-dim}(F) = \sup_{m \geq 0} \left( \dim \{x \in X \mid \text{codim}(F_x) = m\} - m \right),$$

where  $\text{codim}(F_x)$  denotes the codimension of  $\text{Supp}(F_x) \subset \text{Spec}(A)$ . Hence if  $X = \bigsqcup_{\alpha} X_{\alpha}$  is a subanalytic stratification with connected strata and  $F|_{X_{\alpha}}$  is locally constant for any  $\alpha$ , then we have

$$\text{mod-dim}(F) = \sup \{ \dim X_{\alpha} - \text{codim}(F_{x_{\alpha}}) \mid F|_{X_{\alpha}} \neq 0 \},$$

where  $x_{\alpha}$  is a point of  $X_{\alpha}$ . We understand  $\text{mod-dim } 0 = -\infty$ .

We set

$$(5.5) \quad \begin{aligned} ^{1/2}_{\text{KS}} D_{\mathbb{R}-c}^{\leq c}(A_X) &= \{ K \in D_{\mathbb{R}-c}^b(A_X) \mid \text{mod-dim}(H^i(K)) \leq -2(i - c) \text{ for any } i \}, \\ ^{1/2}_{\text{KS}} D_{\mathbb{R}-c}^{\geq c}(A_X) &= \{ K \in D_{\mathbb{R}-c}^b(A_X) \mid D_X K \in ^{1/2}_{\text{KS}} D_{\mathbb{R}-c}^{\leq -c}(A_X) \}. \end{aligned}$$

Note that, when  $A$  is a field, they coincide with  $^{1/2}_{\text{KS}} D_{\mathbb{R}-c}^{\leq c}(A_X)$  and  $^{1/2}_{\text{KS}} D_{\mathbb{R}-c}^{\geq c}(A_X)$ .

**Lemma 5.3.** *Let  $K \in D_{\mathbb{R}-c}^b(A_X)$  and  $c \in \mathbb{R}$ . Let  $X = \bigsqcup_{\alpha} X_{\alpha}$  be a subanalytic stratification such that  $K|_{X_{\alpha}}$  has locally constant cohomologies. Then the following conditions are equivalent;*

- (a)  $K \in ^{1/2}_{\text{KS}} D_{\mathbb{R}-c}^{\leq c}(A_X)$ ,
- (b)  $\dim \{x \in X \mid K_x \notin ^{1/2}_{\text{KS}} D_{\text{coh}}^{\leq c-k/2}(A)\} < k$  for any  $k \in \mathbb{Z}$ ,
- (c)  $K_x \in ^{1/2}_{\text{KS}} D_{\text{coh}}^{\leq c - \dim X_{\alpha}/2}(A)$  for any  $\alpha$  and  $x \in X_{\alpha}$ .

*Proof.* (a) $\Leftrightarrow$ (c) It is obvious that  $K \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_X)$  if and only if

$$\dim X_\alpha - \operatorname{codim} \operatorname{Supp}(H^i(K)_x) \leq -2(i - c) \text{ for any } \alpha, x \in X_\alpha \text{ and } i \in \mathbb{Z}.$$

The last condition is equivalent to

$$\operatorname{codim} \operatorname{Supp}(H^i(K_x)) \geq 2(i - c + \dim X_\alpha/2),$$

or equivalently  $K_x \in {}^{1/2}\mathcal{D}_{\operatorname{coh}}^{\leq c - \dim X_\alpha/2}(A)$ .

(b) $\Leftrightarrow$ (c) (b) is equivalent to the condition:

$$\text{for any } x \in X_\alpha, K_x \notin {}^{1/2}\mathcal{D}_{\operatorname{coh}}^{\leq c - k/2}(A) \text{ implies } \dim X_\alpha < k,$$

which is equivalent to the condition:

$$\text{for any } x \in X_\alpha, \dim X_\alpha \geq k \text{ implies } K_x \in {}^{1/2}\mathcal{D}_{\operatorname{coh}}^{\leq c - k/2}(A).$$

It is obviously equivalent to (c).  $\square$

**Lemma 5.4.** *Let  $K \in \mathcal{D}_{\mathbb{R}\text{-c}}^b(A_X)$  and  $c \in \mathbb{R}$ . Let  $X = \bigsqcup_\alpha X_\alpha$  be a subanalytic stratification such that  $(\mathcal{D}_X K)|_{X_\alpha}$  has locally constant cohomologies. Then the following conditions are equivalent:*

- (a)  $K \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c}(A_X)$ ,
- (b) for any  $c' \in \mathbb{R}$  and  $M \in {}^{1/2}\mathcal{D}_{\operatorname{coh}}^{\leq c'}(A)$ , we have

$$\operatorname{R}\mathcal{H}om_A(M_X, K) \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c - c'}(A_X),$$

- (c)  $\operatorname{R}\Gamma_Z(K)_x \in {}^{1/2}\mathcal{D}_{\operatorname{coh}}^{\geq c - \dim Z/2}(A)$  for any closed subanalytic set  $Z$  and  $x \in Z$ ,
- (d)  $(\operatorname{R}\Gamma_{X_\alpha} K)_x \in {}^{1/2}\mathcal{D}_{\operatorname{coh}}^{\geq c - \dim X_\alpha/2}(A)$  for any  $\alpha$  and  $x \in X_\alpha$ ,
- (e)  $\dim \left\{ x \in X \mid (\operatorname{R}\Gamma_{\{x\}} K)_x \notin {}^{1/2}\mathcal{D}_{\operatorname{coh}}^{\geq c + k/2}(A) \right\} < k$  for any  $k \in \mathbb{Z}_{\geq 0}$ .

*Proof.* Let  $i_\alpha: X_\alpha \rightarrow X$  be the inclusion.

(a) $\Leftrightarrow$ (d) By (a) $\Leftrightarrow$ (c) in the preceding lemma, condition (a) is equivalent to the condition:

$$(\mathcal{D}_X K)_x \in {}^{1/2}\mathcal{D}_{\operatorname{coh}}^{\leq -c - \dim X_\alpha/2}(A) \text{ for any } \alpha \text{ and } x \in X_\alpha.$$

On the other hand, we have  $i_\alpha^{-1}\mathcal{D}_X K \simeq \mathcal{D}_{X_\alpha} i_\alpha^! K$ . Hence  $i_\alpha^! K$  has locally constant cohomologies. Since

$$(\mathcal{D}_X K)_x \simeq (\mathcal{D}_{X_\alpha} i_\alpha^! K)_x \simeq \operatorname{RHom}_A((i_\alpha^! K)_x, A)[\dim X_\alpha],$$

the above condition is equivalent to

$$\operatorname{RHom}_A((i_\alpha^! K)_x, A) \in {}^{1/2}\mathcal{D}_{\operatorname{coh}}^{\leq -c + \dim X_\alpha/2}(A),$$

which is again equivalent to  $(i_\alpha^! K)_x \in {}^{1/2}\mathcal{D}_{\operatorname{coh}}^{\geq c - \dim X_\alpha/2}(A)$ .

(a) $\Leftrightarrow$ (e) (a) is equivalent to  $D_X K \in {}^{1/2}D_{\mathbb{R}-c}^{\leq -c}(\mathbf{k}_X)$ . By the preceding lemma, it is equivalent to the condition:  $\dim \left\{ x \in X \mid (D_X K)_x \notin {}^{1/2}D_{\text{coh}}^{\leq -c-k/2}(A) \right\} < k$  for any  $k \in \mathbb{Z}_{\geq 0}$ . Since  $(D_X K)_x \simeq D_A((R\Gamma_{\{x\}} K)_x)$ , the condition  $(D_X K)_x \notin {}^{1/2}D_{\text{coh}}^{\leq -c-k/2}(A)$  is equivalent to  $(R\Gamma_{\{x\}} K)_x \notin {}^{1/2}D_{\text{coh}}^{\geq c+k/2}(A)$ .

(d) $\Leftrightarrow$ (b) Condition (d) is equivalent to

$$(5.6) \quad \text{RHom}_{\alpha} \left( M, (R\Gamma_{X_{\alpha}} K)_x \right) \in D_{\text{coh}}^{\geq c - \dim X_{\alpha}/2 - c'}(A) \text{ for any } M \in {}^{1/2}D_{\text{coh}}^{\leq c'}(A),$$

$\alpha$  and  $x \in X_{\alpha}$ .

Since  $\text{RHom}(M, (R\Gamma_{X_{\alpha}} K)_x) \simeq (R\Gamma_{X_{\alpha}} R\mathcal{H}om_A(M_X, K))_x$ , the last condition (5.6) is equivalent to (b) by Lemma 5.2.

(c) $\Rightarrow$ (d) is obvious.

(b) $\Rightarrow$ (c) For any  $c' \in \mathbb{R}$  and  $M \in {}^{1/2}D_{\text{coh}}^{\leq c'}(A)$ , we have  $(R\Gamma_Z R\mathcal{H}om_A(M_X, K))_x \in D_{\text{coh}}^{\geq c - c' - \dim Z/2}(A)$ . Since  $\text{RHom}_A(M, (R\Gamma_Z K)_x) \simeq (R\Gamma_Z R\mathcal{H}om_A(M_X, K))_x$ , we have (c)

□

We shall prove the following theorem in several steps.

**Theorem 5.5.**  $(({}^{1/2}D_{\mathbb{R}-c}^{\leq c}(A_X))_{c \in \mathbb{R}}, ({}^{1/2}D_{\mathbb{R}-c}^{\geq c}(A_X))_{c \in \mathbb{R}})$  is a  $t$ -structure on  $D_{\mathbb{R}-c}^b(A_X)$ .

It is obvious that it satisfies conditions (a) and (b) in Definition 1.2. Let us show (c).

**Lemma 5.6.** For  $c \in \mathbb{R}$  and  $K \in {}^{1/2}D_{\mathbb{R}-c}^{\leq c}(A_X)$  and  $L \in {}^{1/2}D_{\mathbb{R}-c}^{\geq c'}(A_X)$ , we have

$$R\mathcal{H}om(K, L) \in D_{\mathbb{R}-c}^{\geq c' - c}(A_X).$$

*Proof.* Let us take a good regular subanalytic filtration of  $X$

$$\emptyset = X_{-1} \subset \cdots \subset X_N = X$$

such that  $K$  and  $L$  have locally constant cohomologies on each  $\overset{\circ}{X}_k := X_k \setminus X_{k-1}$ . We may assume that  $\overset{\circ}{X}_k$  is smooth of dimension  $k$ .

Let  $i_k: \overset{\circ}{X}_k \rightarrow X$  be the inclusion.

Let us first show that

$$(5.7) \quad i_k^! R\mathcal{H}om(K, L) \simeq R\mathcal{H}om(i_k^{-1} K, i_k^! L) \text{ belongs to } D_{\mathbb{R}-c}^{\geq c' - c}(A_{\overset{\circ}{X}_k}).$$

Since  $i_k^{-1} K, i_k^! L$  have locally constant cohomologies,

$$(i_k^! R\mathcal{H}om(K, L))_x \simeq \text{RHom}_A((i_k^{-1} K)_x, (i_k^! L)_x)$$

for any  $x \in \overset{\circ}{X}_k$ . Hence it is enough to show that

$$(5.8) \quad \mathrm{RHom}_A((i_k^{-1}K)_x, (i_k^!L)_x) \in D_{\mathbb{R}\text{-c}}^{\geq c' - c}(A).$$

It follows from Corollary 4.3 and  $(i_k^{-1}K)_x \in {}^{1/2}D_{\mathrm{coh}}^{\leq c - k/2}(A)$  and  $(i_k^!L)_x \in {}^{1/2}D_{\mathrm{coh}}^{\geq c' - k/2}(A)$ .

Now we shall show that

$$\mathrm{R}\Gamma_{X_k} \mathrm{R}\mathcal{H}om(K, L) \in D_{\mathbb{R}\text{-c}}^{\geq c' - c}(A_X)$$

by induction on  $k$ . By the induction hypothesis  $\mathrm{R}\Gamma_{X_{k-1}} \mathrm{R}\mathcal{H}om(K, L) \in D_{\mathbb{R}\text{-c}}^{\geq c' - c}(A_X)$ . We have the distinguished triangle

$$\mathrm{R}\Gamma_{X_{k-1}} \mathrm{R}\mathcal{H}om(K, L) \longrightarrow \mathrm{R}\Gamma_{X_k} \mathrm{R}\mathcal{H}om(K, L) \longrightarrow \mathrm{R}\Gamma_{\overset{\circ}{X}_k} \mathrm{R}\mathcal{H}om(K, L) \xrightarrow{+1}$$

Since  $\mathrm{R}\Gamma_{\overset{\circ}{X}_k} \mathrm{R}\mathcal{H}om(K, L) \simeq \mathrm{R}i_k^* i_k^! \mathrm{R}\mathcal{H}om(K, L)$  belongs to  $D_{\mathbb{R}\text{-c}}^{\geq c' - c}(A_X)$ , we obtain  $\mathrm{R}\Gamma_{X_k} \mathrm{R}\mathcal{H}om(K, L) \in D_{\mathbb{R}\text{-c}}^{\geq c' - c}(A_X)$ .  $\square$

Now we shall show Definition 1.2 (d) in a special case.

**Lemma 5.7.** *Let us assume that  $X$  is a smooth subanalytic space, and  $c \in \mathbb{R}$ . Let  $K \in D_{\mathbb{R}\text{-c}}^b(A_X)$  and assume that  $K$  has locally constant cohomologies. Then there exists a distinguished triangle*

$$K' \longrightarrow K \longrightarrow K'' \xrightarrow{+1}$$

with  $K' \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{\leq c}(A_X)$  and  $K'' \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{\geq c}(A_X)$ . Moreover  $K'$  and  $K''$  have locally constant cohomologies.

*Proof.* Let us show it in three steps.

(i) There exists locally such a distinguished triangle.

Indeed, for any  $x \in X$ , there exist an open neighborhood  $U$  of  $x$  and  $M \in D_{\mathrm{coh}}^b(A)$  such that  $K|_U \simeq M_U$ . Take a distinguished triangle  $M' \rightarrow M \rightarrow M'' \xrightarrow{+1}$  such that  $M' \in {}^{1/2}D_{\mathrm{coh}}^{\leq c - \dim X/2}(A)$  and  $M'' \in {}^{1/2}D_{\mathrm{coh}}^{\geq c - \dim X/2}(A)$ . Then  $M'_U \rightarrow M_U \rightarrow M''_U \xrightarrow{+1}$  gives a desired distinguished triangle.

(ii) If  $U_i$  is an open subset of  $X$  and  $K'_i \rightarrow K|_{U_i} \rightarrow K''_i \xrightarrow{+1}$  is a distinguished triangle with  $K'_i \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{\leq c}(A_{U_i})$  and  $K''_i \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{\geq c}(A_{U_i})$  ( $i = 1, 2$ ), then there exists a distinguished triangle  $K' \rightarrow K|_{U_1 \cup U_2} \rightarrow K'' \xrightarrow{+1}$  with  $K' \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{\leq c}(A_{U_1 \cup U_2})$  and  $K'' \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{\geq c}(A_{U_1 \cup U_2})$ .

Indeed, by the uniqueness of such a distinguished triangle, we have  $K'_1|_{U_1 \cap U_2} \simeq K'_2|_{U_1 \cap U_2}$ . Denote it by  $K_0 \in D^b(A_{U_1 \cap U_2})$ . Let  $i_0: U_1 \cap U_2 \rightarrow U_1 \cup U_2$  and  $i_k: U_k \rightarrow U_1 \cup U_2$  ( $k = 1, 2$ ) be the open inclusions. Then embed a morphism  $i_{0!}K_0 \rightarrow i_{1!}K'_1 \oplus i_{2!}K'_2$  into a distinguished triangle

$$i_{0!}K_0 \rightarrow i_{1!}K'_1 \oplus i_{2!}K'_2 \rightarrow K' \xrightarrow{+1}.$$

Then  $K'|_{U_k} \simeq K'_k$ . Since the composition  $i_0!K_0 \rightarrow i_1!K'_1 \oplus i_2!K'_2 \rightarrow K|_{U_1 \cup U_2}$  vanishes, the morphism  $i_1!K'_1 \oplus i_2!K'_2 \rightarrow K|_{U_1 \cup U_2}$  factors through  $K'$ . Hence, there exists a morphism  $K' \rightarrow K|_{U_1 \cup U_2}$  which extends  $K'_i \rightarrow K|_{U_i}$  ( $i = 1, 2$ ). Then, embedding this morphism into a distinguished triangle  $K' \rightarrow K|_{U_1 \cup U_2} \rightarrow K'' \xrightarrow{+1}$ , we obtain the desired distinguished triangle.

(iii) By (i) and (ii), there exist an increasing sequence of open subsets  $\{U_n\}_{n \in \mathbb{Z}_{\geq 0}}$  with  $X = \bigcup_{n \in \mathbb{Z}_{\geq 0}} U_n$  and a distinguished triangle  $K_n \rightarrow K|_{U_n} \rightarrow K''_n \xrightarrow{+1}$  with  $K'_n \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{\leq c}(A_{U_n})$  and  $K''_n \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{> c}(A_{U_n})$ . Let  $i_n: U_n \rightarrow X$  be the inclusion. By the uniqueness of such distinguished triangles, we have  $K'_{n+1}|_{U_n} \simeq K'_n$ . Hence, we have  $\beta_n: i_n!K'_n \rightarrow i_{n+1}!K'_{n+1}$ . Let  $K'$  be the hocolim of the inductive system  $\{i_n!K'_n\}_{n \in \mathbb{Z}_{\geq 0}}$ , that is, the third term of a distinguished triangle

$$\bigoplus_{n \in \mathbb{Z}_{\geq 0}} i_n!K'_n \xrightarrow{f} \bigoplus_{n \in \mathbb{Z}_{\geq 0}} i_n!K'_n \rightarrow K' \xrightarrow{+1}.$$

Here  $f$  is given such that the following diagram commutes for any  $a \in \mathbb{Z}_{\geq 0}$ :

$$\begin{array}{ccc} i_a!K'_a & \xrightarrow{\text{id}_{i_a!K'_a} \oplus (-\beta_a)} & i_a!K'_a \oplus i_{a+1}!K'_{a+1} \\ \downarrow & & \downarrow \\ \bigoplus_{n \in \mathbb{Z}_{\geq 0}} i_n!K'_n & \xrightarrow{f} & \bigoplus_{n \in \mathbb{Z}_{\geq 0}} i_n!K'_n. \end{array}$$

Then  $K'|_{U_n} \simeq K'_n$ . Since the composition

$$\bigoplus_{n \in \mathbb{Z}_{\geq 0}} i_n!K'_n \xrightarrow{f} \bigoplus_{n \in \mathbb{Z}_{\geq 0}} i_n!K'_n \rightarrow K$$

vanishes, the morphism  $\bigoplus_{n \in \mathbb{Z}_{\geq 0}} i_n!K'_n \rightarrow K$  factors through  $K'$ . Hence there is a morphism  $K' \rightarrow K$  which extends  $i_n!K'_n \rightarrow K$ . Hence, embedding this morphism into a distinguished triangle

$$K' \rightarrow K \rightarrow K'' \xrightarrow{+1},$$

we obtain the desired distinguished triangle.  $\square$

Finally we shall complete the proof of Definition 1.2 (d).

**Lemma 5.8.** *Let  $K \in D_{\mathbb{R}\text{-c}}^b(A_X)$  and  $c \in \mathbb{R}$ . Then there exists a distinguished triangle*

$$K' \rightarrow K \rightarrow K'' \xrightarrow{+1}$$

*with  $K' \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{\leq c}(A_X)$  and  $K'' \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{> c}(A_X)$ .*

*Proof.* Let us take a good regular subanalytic filtration of  $X$

$$\emptyset = X_{-1} \subset \cdots \subset X_N = X$$

such that  $K$  has locally constant cohomologies on each  $\overset{\circ}{X}_k := X_k \setminus X_{k-1}$ . We may assume that  $\overset{\circ}{X}_k$  is a smooth subanalytic space of dimension  $k$ . We shall prove that

$$(5.2.9)_k \quad \begin{cases} \text{there exists a distinguished triangle} \\ K' \longrightarrow K|_{X \setminus X_k} \longrightarrow K'' \xrightarrow{+1} \\ \text{with } K \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_{X \setminus X_k}) \text{ and } K'' \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{> c}(A_X). \text{ Moreover, } K'|_{\overset{\circ}{X}_j} \text{ and} \\ K''|_{\overset{\circ}{X}_j} \text{ have locally constant cohomologies for } j > k. \end{cases}$$

by the descending induction on  $k$ .

Assuming (5.2.9) $_k$ , we shall show (5.2.9) $_{k-1}$ . Let  $K' \longrightarrow K|_{X \setminus X_k} \longrightarrow K'' \xrightarrow{+1}$  be a distinguished triangle as in (5.2.9) $_k$ . Let  $j: X \setminus X_k \rightarrow X \setminus X_{k-1}$  be the open embedding and  $i: \overset{\circ}{X}_k \rightarrow X \setminus X_{k-1}$  the closed embedding. The morphism  $K' \rightarrow K|_{X \setminus X_k}$  induces  $j_! K' \rightarrow K|_{X \setminus X_{k-1}}$ . We embed it into a distinguished triangle in  $\mathcal{D}_{\mathbb{R}\text{-c}}^b(A_{X \setminus X_{k-1}})$

$$j_! K' \rightarrow K|_{X \setminus X_{k-1}} \rightarrow L \xrightarrow{+1}.$$

By Lemma 5.7, there exists a distinguished triangle

$$(5.10) \quad L' \longrightarrow i^! L \longrightarrow L'' \xrightarrow{+1}$$

with  $L' \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_{\overset{\circ}{X}_k})$  and  $L'' \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{> c}(A_{\overset{\circ}{X}_k})$ . We embed the composition  $i_! L' \rightarrow i_! i^! L \rightarrow L$  into a distinguished triangle

$$(5.11) \quad i_! L' \rightarrow L \rightarrow \tilde{K}'' \xrightarrow{+1}$$

Finally we embed the composition  $K|_{X \setminus X_{k-1}} \rightarrow L \rightarrow \tilde{K}''$  into a distinguished triangle

$$\tilde{K}' \rightarrow K|_{X \setminus X_{k-1}} \rightarrow \tilde{K}'' \xrightarrow{+1}.$$

Let us show that

$$\tilde{K}' \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_{X \setminus X_{k-1}}) \text{ and } \tilde{K}'' \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{> c}(A_{X \setminus X_{k-1}}).$$

By the construction, we have  $\tilde{K}''|_{X \setminus X_k} \simeq L|_{X \setminus X_k} \simeq K''$  and  $\tilde{K}'|_{X \setminus X_k} \simeq K'$ . Hence it is enough to show that  $i^{-1} \tilde{K}' \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_{\overset{\circ}{X}_k})$  and  $i^! \tilde{K}'' \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{> c}(A_{\overset{\circ}{X}_k})$ . Applying the functor  $i^!$  to (5.11), we obtain a distinguished triangle

$$L' \longrightarrow i^! L \longrightarrow i^! \tilde{K}'' \xrightarrow{+1}.$$

By the distinguished triangle (5.10), we have  $i^! \tilde{K}'' \simeq L'' \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{> c}(A_{\overset{\circ}{X}_k})$ .



By the octahedral axiom for triangulated category, we have a diagram

$$\begin{array}{ccccc}
 & & \tilde{K}' & & \\
 & \swarrow \cdots & & \searrow \cdots & \\
 j_! K' & \xleftarrow{+1} & & i_! L' & \\
 \downarrow & \swarrow +1 & & \searrow +1 & \uparrow +1 \\
 K|_{X \setminus X_{k-1}} & \xrightarrow{\quad} & \tilde{K}'' & & \\
 & \searrow & & \swarrow & \\
 & & L & & 
 \end{array}$$

and a distinguished triangle

$$j_! K' \longrightarrow \tilde{K}' \longrightarrow i_! L' \xrightarrow{+1}.$$

It implies  $i^{-1} \tilde{K}' \simeq L' \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{\leq c}(A_{\circ_{X_k}})$ . □

This completes the proof of Theorem 5.5.

Recall the full subcategory of  $D_{\mathbb{R}\text{-c}}^b(A_X)$ :

$${}^{1/2}D_{\mathbb{R}\text{-c}}^{[a,b]}(A_X) := {}^{1/2}D_{\mathbb{R}\text{-c}}^{\leq b}(A_X) \cap {}^{1/2}D_{\mathbb{R}\text{-c}}^{\geq a}(A_X)$$

for  $a \leq b$ .

**Proposition 5.9.** *Assume that  $a, b \in \mathbb{R}$  satisfy  $a \leq b < a + 1$ . Then  $X \supset U \mapsto {}^{1/2}D_{\mathbb{R}\text{-c}}^{[a,b]}((A_U))$  is a stack on  $X$ .*

*Proof.* (i) Let  $K, L \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{[a,b]}(A_X)$ . Since  $R\mathcal{H}om_A(K, L) \in D_{\mathbb{R}\text{-c}}^{\geq a-b}(A_X) = D_{\mathbb{R}\text{-c}}^{\geq 0}(A_X)$ , the presheaf  $U \mapsto \text{Hom}_{{}^{1/2}D_{\mathbb{R}\text{-c}}^{[a,b]}(A_U)}(K|_U, L|_U) \simeq \Gamma(U; H^0(R\mathcal{H}om_A(K, L)))$  is a sheaf.

Hence,  $U \mapsto {}^{1/2}D_{\mathbb{R}\text{-c}}^{[a,b]}(A_U)$  is a separated prestack on  $X$ .

(ii) Let us show the following statement:

Let  $U_1$  and  $U_2$  be open subsets of  $X$  such that  $X = U_1 \cup U_2$ , and  $K_k \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{[a,b]}(A_{U_k})$  ( $k = 1, 2$ ). Assume that  $K_1|_{U_1 \cap U_2} \simeq K_2|_{U_1 \cap U_2}$ . Then there exists  $K \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{[a,b]}(A_X)$  such that  $K|_{U_k} \simeq K_k$  ( $k = 1, 2$ ).

Set  $U_0 = U_1 \cap U_2$  and  $K_0 = K_1|_{U_1 \cap U_2} \simeq K_2|_{U_1 \cap U_2} \in {}^{1/2}D_{\mathbb{R}\text{-c}}^{[a,b]}(A_{U_0})$ . Let  $j_k: U_k \rightarrow X$  be the open inclusion ( $k = 0, 1, 2$ ). Then we have  $\beta_k: j_{0!}(K_0) \rightarrow j_{k!}K_k$  ( $k = 1, 2$ ). We embed the morphism  $(\beta_1, \beta_2): j_{0!}(K_0) \rightarrow j_{1!}K_1 \oplus j_{2!}K_2$  into a distinguished triangle

$$j_{0!}(K_0) \longrightarrow j_{1!}K_1 \oplus j_{2!}K_2 \longrightarrow K \xrightarrow{+1}.$$

Then  $K$  satisfies the desired condition.

(iii) Let us show the following statement:

Let  $\{U_n\}_{n \in \mathbb{Z}_{\geq 0}}$  be an increasing sequence of open subsets of  $X$  such that  $X = \bigcup_{n \in \mathbb{Z}_{\geq 0}} U_n$ . Let  $K_n \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{[a,b]}(A_{U_n})$  ( $n \in \mathbb{Z}_{\geq 0}$ ) and  $K_{n+1}|_{U_n} \simeq K_n$ . Then there exists  $K \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{[a,b]}(A_X)$  such that  $K|_{U_n} \simeq K_n$  ( $n \in \mathbb{Z}_{\geq 0}$ ).

The proof is similar to proof of Lemma 5.7. Let  $j_n: U_n \rightarrow X$  be the open inclusion, and let  $j_{n!}K_n \rightarrow j_{n+1!}K_{n+1}$  be the morphism induced by the isomorphism  $K_{n+1}|_{U_n} \simeq K_n$ . Let  $K$  be a hocolim of the inductive system  $\{j_{n!}K_n\}_{n \in \mathbb{Z}_{\geq 0}}$ . Then  $K \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{[a,b]}(A_X)$  satisfies the desired condition.

(iv) By (i)–(iii), we conclude that  $U \mapsto {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{[a,b]}(A_U)$  is a stack on  $X$ .  $\square$

**Proposition 5.10.** *Let  $f: X \rightarrow Y$  be a morphism of subanalytic spaces, and  $d \in \mathbb{Z}_{\geq 0}$ . Assume that  $\dim f^{-1}(y) \leq d$  for any  $y \in Y$ . Then*

- (i) *If  $G \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_Y)$ , then  $f^{-1}G \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c+d/2}(A_X)$ .*
- (ii) *If  $G \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c}(A_Y)$ , then  $f^!G \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c-d/2}(A_X)$ .*
- (iii) *If  $F \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c}(A_X)$  and  $Rf_*F \in \mathcal{D}_{\mathbb{R}\text{-c}}^b(A_Y)$ , then  $Rf_*F \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c-d/2}(A_Y)$ .*
- (iv) *If  $F \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_X)$  and  $Rf_!F \in \mathcal{D}_{\mathbb{R}\text{-c}}^b(A_Y)$ , then  $Rf_!F \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c+d/2}(A_Y)$ .*

*Proof.* (i) Assume  $G \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_Y)$ . Then

$$\begin{aligned} & \dim \left\{ x \in X \mid (f^{-1}G)_x \notin {}^{1/2}\mathcal{D}_{\text{coh}}^{\leq c+d/2-k/2}(A) \right\} \\ &= \dim f^{-1} \left( \left\{ y \in Y \mid G_y \notin {}^{1/2}\mathcal{D}_{\text{coh}}^{\leq c+d/2-k/2}(A) \right\} \right) \\ &\leq \dim \left\{ y \in Y \mid G_y \notin {}^{1/2}\mathcal{D}_{\text{coh}}^{\leq c+d/2-k/2}(A) \right\} + d \\ &< (k-d) + d = k. \end{aligned}$$

(ii) follows from (i) by the duality.

(iii) For any  $G \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c-d/2}(A_Y)$ , then

$$\text{Hom}_{\mathcal{D}_{\mathbb{R}\text{-c}}^b(A_Y)}(G, Rf_*F) \simeq \text{Hom}_{\mathcal{D}_{\mathbb{R}\text{-c}}^b(A_X)}(f^{-1}G, F)$$

vanishes because  $f^{-1}G \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_X)$  by (i). Hence  $Rf_*F \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c-d/2}(A_Y)$  by (1.3).

Similarly, (iv) follows from (ii).  $\square$

We shall give relations between the two t-structures:

$$\left( ({}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_X))_{c \in \mathbb{R}}, ({}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c}(A_X))_{c \in \mathbb{R}} \right) \text{ and } \left( ({}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_X))_{c \in \mathbb{R}}, ({}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c}(A_X))_{c \in \mathbb{R}} \right).$$

**Lemma 5.11.** *Let  $K \in \mathcal{D}_{\mathbb{R}\text{-c}}^b(A_X)$  and  $c \in \mathbb{R}$ .*

- (i) *The following conditions are equivalent:*
  - (a)  $K \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_X)$ ,

(b) for any  $c' \in \mathbb{R}$  and  $M \in {}^{1/2}\mathcal{D}_{\text{coh}}^{\geq c'}(A)$ , we have

$$\mathcal{R}\mathcal{H}om_A(K, M \otimes \omega_X) \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c'-c}(A_X).$$

(ii) The following conditions are equivalent:

(a)  $K \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c}(A_X)$ ,

(b) for any  $c' \in \mathbb{R}$  and  $M \in {}^{1/2}\mathcal{D}_{\text{coh}}^{\leq c'}(A)$ , we have

$$\mathcal{R}\mathcal{H}om_A(M_X, K) \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c-c'}(A_X).$$

*Proof.* (ii) is already proved in Lemma 5.4. (i) follows from (ii) because

$$\begin{aligned} \mathcal{R}\mathcal{H}om_A(K, M \otimes \omega_X) &\simeq \mathcal{R}\mathcal{H}om_A((D_X(M \otimes \omega_X), D_X K)) \\ &\simeq \mathcal{R}\mathcal{H}om_A((D_A M)_X, D_X K), \end{aligned}$$

where  $D_A M := \mathcal{R}\mathcal{H}om_A(M, A)$ . □

**Lemma 5.12.** *Let  $X$  and  $Y$  be subanalytic spaces. Let  $K \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c}(A_X)$  and  $L \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c'}(A_Y)$ . Then we have  $K \boxtimes L \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c+c'}(A_{X \times Y})$ .*

*Proof.* Let  $X = \bigsqcup_{\alpha} X_{\alpha}$  and  $Y = \bigsqcup_{\beta} Y_{\beta}$  be good subanalytic stratification such that  $K|_{X_{\alpha}}$  and  $L|_{Y_{\beta}}$  are locally constant. Then we have  $(\mathcal{R}\Gamma_{X_{\alpha}} K)_x \in {}^{1/2}\mathcal{D}_{\text{coh}}^{\geq c - \dim X_{\alpha}/2}(A)$  and  $(\mathcal{R}\Gamma_{Y_{\beta}} L)_y \in {}^{1/2}\mathcal{D}_{\text{coh}}^{\geq c' - \dim Y_{\beta}/2}(A)$  for  $x \in X_{\alpha}$  and  $y \in Y_{\beta}$ . Hence by Proposition 4.4 (iv), we have

$$(\mathcal{R}\Gamma_{X_{\alpha} \times Y_{\beta}}(K \boxtimes L))_{(x,y)} \simeq (\mathcal{R}\Gamma_{X_{\alpha}} K)_x \otimes (\mathcal{R}\Gamma_{Y_{\beta}} L)_y \in \mathcal{D}_{\text{coh}}^{\geq c+c' - \dim(X_{\alpha} \times Y_{\beta})/2}(A).$$

Hence we obtain  $K \boxtimes L \in {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c+c'}(A_{X \times Y})$ . □

**Remark 5.13.** We have

$${}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_X) \subset {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\leq c}(A_X),$$

$${}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c}(A_X) \subset {}^{1/2}\mathcal{D}_{\mathbb{R}\text{-c}}^{\geq c}(A_X).$$

## 6. SELF-DUAL T-STRUCTURE: COMPLEX ANALYTIC VARIETY CASE

**6.1. Middle perversity in the complex case.** Let  $X$  be a complex analytic space. We denote by  $\dim_{\mathbb{C}} X$  the dimension of  $X$ . Hence we have  $\dim_{\mathbb{C}} X = (\dim X_{\mathbb{R}})/2$  where  $X_{\mathbb{R}}$  is the underlying subanalytic space. For a complex submanifold  $Y$  of a complex manifold  $X$ , we denote by  $\text{codim}_{\mathbb{C}} Y$  the codimension of  $Y$  as complex manifolds. We sometimes write  $d_X$  for  $\dim_{\mathbb{C}} X$ .

Let  $\mathcal{D}_{\mathbb{C}\text{-c}}^b(A_X)$  be the bounded derived category of the abelian category of sheaves of  $A$ -modules with  $\mathbb{C}$ -constructible cohomologies. Then it is a full subcategory of

$D_{\mathbb{R}-c}^b(A_X)$  and it is easy to see that the self-dual t-structure on  $D_{\mathbb{R}-c}^b(A_X)$  induces a self-dual t-structures on  $D_{\mathbb{C}-c}^b(A_X)$ . More precisely, if we define

$$\begin{aligned} {}^{1/2}D_{\mathbb{C}-c}^{\leq c}(A_X) &:= D_{\mathbb{C}-c}^b(A_X) \cap {}^{1/2}D_{\mathbb{R}-c}^{\leq c}(A_X) \quad \text{and} \\ {}^{1/2}D_{\mathbb{C}-c}^{\geq c}(A_X) &:= D_{\mathbb{C}-c}^b(A_X) \cap {}^{1/2}D_{\mathbb{R}-c}^{\geq c}(A_X), \end{aligned}$$

then  $((^{1/2}D_{\mathbb{C}-c}^{\leq c}(A_X))_{c \in \mathbb{C}}, (^{1/2}D_{\mathbb{C}-c}^{\geq c}(A_X))_{c \in \mathbb{C}})$  is a t-structure on  $D_{\mathbb{C}-c}^b(A_X)$ .

Similarly, the t-structure  $((^{1/2}_{\text{KS}}D_{\mathbb{R}-c}^{\leq c}(A_X))_{c \in \mathbb{C}}, (^{1/2}_{\text{KS}}D_{\mathbb{R}-c}^{\geq c}(A_X))_{c \in \mathbb{C}})$  induces the t-structure  $((^{1/2}_{\text{KS}}D_{\mathbb{C}-c}^{\leq c}(A_X))_{c \in \mathbb{C}}, (^{1/2}_{\text{KS}}D_{\mathbb{C}-c}^{\geq c}(A_X))_{c \in \mathbb{C}})$  on  $D_{\mathbb{C}-c}^b(A_X)$ .

Note that the t-structure  $(^{1/2}_{\text{KS}}D_{\mathbb{C}-c}^{\leq 0}(A_X), ^{1/2}_{\text{KS}}D_{\mathbb{C}-c}^{\geq 0}(A_X))$  in the original sense is denoted by  $({}^pD_{\mathbb{C}-c}^{\leq 0}(X), {}^pD_{\mathbb{C}-c}^{\geq 0}(X))$  in [5, §10.3].

In [5, §10.3], various properties of  $(^{1/2}_{\text{KS}}D_{\mathbb{C}-c}^{\leq 0}(A_X), ^{1/2}_{\text{KS}}D_{\mathbb{C}-c}^{\geq 0}(A_X))$  are studied. By using Lemma 5.11, we have similar properties for  $((^{1/2}D_{\mathbb{C}-c}^{\leq c}(A_X))_{c \in \mathbb{C}}, (^{1/2}D_{\mathbb{C}-c}^{\geq c}(A_X))_{c \in \mathbb{C}})$  as we explain in the next subsection.

**6.2. Microlocal characterization.** Let us assume that  $X$  is a complex manifold. Let  $K \in D_{\mathbb{C}-c}^b(A_X)$ . Then its microsupport  $\text{SS}(K)$  is a Lagrangian complex analytic subset of the cotangent bundle  $T^*X$  (see [5]).

A point  $p$  of  $\text{SS}(K)$  is called *good* if  $\text{SS}(K)$  is equal to the conormal bundle  $T_Y^*X$  on a neighborhood of  $p$  for some locally closed complex submanifold  $Y$  of  $X$ . The complement of the set of good points of  $\text{SS}(K)$  is a nowhere dense closed complex analytic subset of  $\text{SS}(K)$ . For such a good point  $p$  of  $\text{SS}(K)$ , there exists  $L \in D_{\text{coh}}^b(A)$  such that  $K$  is micro-locally isomorphic to  $L_Y[-\text{codim}_{\mathbb{C}} Y]$  on a neighborhood of  $p$ . We call  $L$  the type of  $K$  at  $p$ . (Note that  $L$  is called the type of  $K$  at  $p$  with shift 0 in [5, §10.3].)

The type can be calculated by the vanishing cycle functor. If we take a holomorphic function such that  $f|_Y = 0$  and  $df(x_0) = p$ , then we have  $\varphi_f(K)_{x_0} \simeq L[-\text{codim}_{\mathbb{C}} Y]$ . Here,  $x_0 \in X$  is the image of  $p$  by the projection  $T^*X \rightarrow X$ , and  $\varphi_f$  is the vanishing cycle functor from  $D_{\mathbb{C}-c}^b(A_X)$  to  $D_{\mathbb{C}-c}^b(A_{f^{-1}(0)})$ . Note that we have an isomorphism

$$\varphi_f(K) \simeq \text{R}\Gamma_{\{x|\text{Re}(f(x)) \geq 0\}}(K)|_{f^{-1}(0)}.$$

The following theorem is proved in [5, §10.3].

**Theorem 6.1** ([5, Theorem 10.3.2]). *Let  $K \in D_{\mathbb{C}-c}^b(A_X)$ . Then the following conditions are equivalent.*

- (a)  $K \in {}^{1/2}_{\text{KS}}D_{\mathbb{C}-c}^{\leq c}(A_X)$  (resp.  $K \in {}^{1/2}_{\text{KS}}D_{\mathbb{C}-c}^{\geq c}(A_X)$ ),
- (b) the type of  $K$  at any good point of  $\text{SS}(K)$  belongs to  $D_{\text{coh}}^{\leq c-d_X}(A)$  (resp. belongs to  $D_{\text{coh}}^{\geq c-d_X}(A)$ ).

As a corollary of this theorem, we can derive the following microlocal characterization of  $((^{1/2}D_{\mathbb{C}-c}^{\leq c}(A_X))_{c \in \mathbb{C}}, (^{1/2}D_{\mathbb{C}-c}^{\geq c}(A_X))_{c \in \mathbb{C}})$ .

**Theorem 6.2.** *Let  $K \in D_{\mathbb{C}-c}^b(A_X)$ . Then the following conditions are equivalent.*

- (a)  $K \in {}^{1/2}D_{\mathbb{C}-c}^{\leq c}(A_X)$  (resp.  $K \in {}^{1/2}D_{\mathbb{C}-c}^{\geq c}(A_X)$ ),
- (b) *the type of  $K$  at any good point of  $\text{SS}(K)$  belongs to  ${}^{1/2}D_{\text{coh}}^{\leq c-d_X}(A)$  (resp. belongs to  ${}^{1/2}D_{\text{coh}}^{\geq c-d_X}(A)$ ).*

*Proof.* Let us assume that  $K \in {}^{1/2}D_{\mathbb{R}-c}^{\geq c}(A_X)$ . Then for any  $M \in {}^{1/2}D_{\text{coh}}^{\leq c'}(A)$ , we have  $R\mathcal{H}om_A(M_X, K) \in {}^{1/2}D_{\text{KS}}^{\geq c-c'}(A_X)$ . Let  $L$  be the type of  $K$  at a good point  $p$  of  $\text{SS}(K)$ . Then,  $R\mathcal{H}om_A(M_X, K)$  has type  $\text{RHom}_A(M, L)$  at  $p$ . Hence, the preceding theorem implies  $\text{RHom}_A(M, L) \in D_{\text{coh}}^{\geq c-c'-d_X}(A)$ . Since this holds for any  $M \in {}^{1/2}D_{\text{coh}}^{\leq c'}(A)$ , we conclude  $L \in {}^{1/2}D_{\text{coh}}^{\geq c-d_X}(A)$ . The converse can be proved similarly.

The case of  ${}^{1/2}D_{\mathbb{C}-c}^{\leq c}(A_X)$  can be derived from the case of  ${}^{1/2}D_{\mathbb{C}-c}^{\geq c}(A_X)$  by duality. Condition  $K \in {}^{1/2}D_{\mathbb{C}-c}^{\leq c}(A_X)$  is equivalent to  $D_X(K) \in {}^{1/2}D_{\mathbb{C}-c}^{\geq -c}(A_X)$ . Let  $L$  be the type of  $K$  at a good point  $p$  of  $\text{SS}(K)$ . Then,  $D_X(K)$  has type  $D_A(L)[2d_X]$  at  $p$ . Then it is enough to notice that  $D_A(L)[2d_X] \in {}^{1/2}D_{\text{coh}}^{\leq -c-d_X}(A)$ , if and only if  $L \in {}^{1/2}D_{\text{coh}}^{\geq c-d_X}(A)$ .  $\square$

The following proposition can be proved similarly.

**Proposition 6.3.** *Let  $Y$  be a closed complex submanifold of a complex manifold  $X$ . Then we have*

- (i) *The functor  $\nu_Y: D_{\mathbb{C}-c}^b(A_X) \rightarrow D_{\mathbb{C}-c}^b(A_{T_Y X})$  sends  ${}^{1/2}D_{\mathbb{C}-c}^{\leq c}(A_X)$  to  ${}^{1/2}D_{\mathbb{C}-c}^{\leq c}(A_{T_Y X})$  and  ${}^{1/2}D_{\mathbb{C}-c}^{\geq c}(A_X)$  to  ${}^{1/2}D_{\mathbb{C}-c}^{\geq c}(A_{T_Y X})$ ,*
- (ii) *The microlocalization functor  $\mu_Y: D_{\mathbb{C}-c}^b(A_X) \rightarrow D_{\mathbb{C}-c}^b(A_{T_Y^* X})$  sends  ${}^{1/2}D_{\mathbb{C}-c}^{\leq c}(A_X)$  to  ${}^{1/2}D_{\mathbb{C}-c}^{\leq c+\text{codim}_{\mathbb{C}} Y}(A_{T_Y^* X})$  and  ${}^{1/2}D_{\mathbb{C}-c}^{\geq c}(A_X)$  to  ${}^{1/2}D_{\mathbb{C}-c}^{\geq c+\text{codim}_{\mathbb{C}} Y}(A_{T_Y^* X})$ .*

*Proof.* Since the proof is similar, we show only (ii). Let  $K \in {}^{1/2}D_{\mathbb{C}-c}^{\geq c}(A_X)$ . Then, for any  $M \in {}^{1/2}D_{\text{coh}}^{\leq c'}(A)$ , we have  $R\mathcal{H}om_A(M_X, K) \in {}^{1/2}D_{\text{KS}}^{\geq c-c'}(A_X)$ . Hence [5, Prop. 10.3.19] implies that  $\mu_Y(R\mathcal{H}om_A(M_X, K)) \in {}^{1/2}D_{\mathbb{C}-c}^{\geq c-c'+\text{codim}_{\mathbb{C}} Y}(A_{T_Y^* X})$ . Since we have

$$R\mathcal{H}om_A(M_{T_Y^* X}, \mu_Y K) \simeq \mu_Y(R\mathcal{H}om_A(M, K)),$$

we obtain  $\mu_Y K \in {}^{1/2}D_{\mathbb{C}-c}^{\geq c+\text{codim}_{\mathbb{C}} Y}(A_{T_Y^* X})$ .

Assume now that  $K \in {}^{1/2}D_{\mathbb{C}-c}^{\leq c}(A_X)$ . Then we have  $D_X K \in {}^{1/2}D_{\mathbb{C}-c}^{\geq -c}(A_X)$ . Since [5, Prop. 8.4.13] implies  $D_{T_Y^* X}(\mu_Y K) \simeq (\mu_Y D_X K)^a[2\text{codim}_{\mathbb{C}} Y]$ , we obtain

$$D_{T_Y^* X}(\mu_Y K) \in {}^{1/2}D_{\mathbb{C}-c}^{\geq -c-\text{codim}_{\mathbb{C}} Y}(A_{T_Y^* X}).$$

Hence  $\mu_Y K \in {}^{1/2}D_{\mathbb{C}-c}^{\leq c+\text{codim}_{\mathbb{C}} Y}(A_{T_Y^* X})$ .  $\square$

The following theorem is proved in [5, §10.3].

**Theorem 6.4** ([5, Corollary 10.3.20]). *Let  $K \in {}^{1/2}D_{\mathbb{C}-c}^{\leq c}(A_X)$  and  $L \in {}^{1/2}D_{\mathbb{C}-c}^{\geq c'}(A_X)$ . Then  $\mu_{\text{hom}}(K, L) \in {}^{1/2}D_{\mathbb{C}-c}^{\geq c'-c+d_X}(A_{T_Y^* X})$ .*

As its corollary we obtain the following result.

**Theorem 6.5.** *Let  $K \in D_{\mathbb{C}-c}^b(A_X)$  and  $L \in D_{\mathbb{C}-c}^b(A_X)$ .*

(i) *If  $K \in {}^{1/2}D_{\mathbb{C}-c}^{\leq c}(A_X)$  and  $L \in {}^{1/2}D_{\mathbb{C}-c}^{\geq c'}(A_X)$ , then we have*

$$\mu hom(K, L) \in {}^{1/2}_{\text{KS}} D_{\mathbb{C}-c}^{\geq c'-c+d_X}(A_{T^*X}).$$

(ii) *If  $K \in {}^{1/2}_{\text{KS}} D_{\mathbb{C}-c}^{\leq c}(A_X)$  and  $L \in {}^{1/2}D_{\mathbb{C}-c}^{\geq c'}(A_X)$ , then we have*

$$\mu hom(K, L) \in {}^{1/2}D_{\mathbb{C}-c}^{\geq c'-c+d_X}(A_{T^*X}).$$

*Proof.* (i) By Lemma 5.12, we have  $L \overset{L}{\boxtimes} D_X K \in {}^{1/2}_{\text{KS}} D_{\mathbb{R}-c}^{\geq c'-c}(A_X)$ . Let  $\Delta_X$  be the diagonal set of  $X \times X$ . Then we have  $\mu hom(K, L) = \mu_{\Delta_X}(L \overset{L}{\boxtimes} D_X K) \in {}^{1/2}_{\text{KS}} D_{\mathbb{C}-c}^{\geq c'-c+d_X}(A_X)$  by [5, Proposition 10.3.19].

(ii) For any  $M \in {}^{1/2}D_{\text{coh}}^{\leq c''}(A)$ , we have  $R\mathcal{H}om(M_X, L) \in {}^{1/2}_{\text{KS}} D_{\mathbb{C}-c}^{\geq c'-c''}(A_X)$ . Hence

$$R\mathcal{H}om(M_{T^*X}, \mu hom(K, L)) \simeq \mu hom(K, R\mathcal{H}om(M_X, L))$$

belongs to  ${}^{1/2}_{\text{KS}} D_{\mathbb{C}-c}^{\geq c'-c''-c+d_X}(A_{T^*X})$  by Theorem 6.4. Hence we conclude  $\mu hom(K, L) \in {}^{1/2}D_{\mathbb{C}-c}^{\geq c'-c+d_X}(A_{T^*X})$  by Lemma 5.11.  $\square$

**Example 6.6.** Assume that 2 acts injectively on  $A$ . Let  $M$  be a finitely generated projective  $A$ -module. Let  $X = \mathbb{C}^3$  and  $S = \{(x, y, z) \in X \mid x^2 + y^2 + z^2 = 0\}$ . Let  $j: X \setminus \{0\} \rightarrow X$  be the inclusion. Since  $S \setminus \{0\}$  is homeomorphic to the product of  $\mathbb{R}$  and the 3-dimensional real projective space  $\mathbb{P}^3(\mathbb{R})$ , we have

$$(Rj_*j^{-1}(M_S))_0 \simeq R\Gamma(S \setminus \{0\}; M_S) \simeq M \oplus (M/2M)[-2] \oplus M[-3],$$

and  $R\Gamma_{\{0\}}(M_S)_0 \simeq (M/2M)[-3] \oplus M[-4]$ . Hence we have

$$M_S \in {}^{1/2}D_{\mathbb{C}-c}^2(A_X),$$

and a distinguished triangle

$$M_0[-1] \rightarrow Rj_*j^{-1}(M_S) \rightarrow M_S \xrightarrow{+1}.$$

Hence we obtain

$$\begin{aligned} Rj_*j^{-1}(M_S) &\in {}^{1/2}D_{\mathbb{C}-c}^{[1,2]}(A_X), \\ {}^{1/2}\tau^{\geq 2}Rj_*j^{-1}(M_S) &\simeq M_S, \\ {}^{1/2}\tau^{< 2}Rj_*j^{-1}(M_S) &\simeq M_0[-1] \in {}^{1/2}D_{\mathbb{C}-c}^1(A_X). \end{aligned}$$

Here  ${}^{1/2}\tau$  denotes the truncation functor of the t-structure  ${}^{1/2}D_{\mathbb{C}-c}^b(A_X)$ .

By the duality, we have

$$Rj_*j^{-1}(M_S) \in {}^{1/2}D_{\mathbb{C}-c}^{[2,3]}(A_X),$$

$${}^{1/2}\tau^{>2}Rj_*j^{-1}(M_S) \simeq M_0[-3] \in {}^{1/2}D_{\mathbb{C}\text{-c}}^3(A_X).$$

Hence we obtain a distinguished triangle

$${}^{1/2}\tau^{\leq 2}Rj_*j^{-1}(M_S) \rightarrow Rj_*j^{-1}(M_S) \rightarrow M_0[-3] \xrightarrow{+1}.$$

The canonical morphism  $Rj_!j^{-1}(M_S) \rightarrow Rj_*j^{-1}(M_S)$  decomposes as

$$\begin{array}{ccc} Rj_!j^{-1}(M_S) & \longrightarrow & Rj_*j^{-1}(M_S) \\ \downarrow & & \uparrow \\ M_S & \longrightarrow & {}^{1/2}\tau^{\leq 2}Rj_*j^{-1}(M_S) \end{array}$$

and the bottom arrow is embedded into a distinguished triangle

$$M_S \rightarrow {}^{1/2}\tau^{\leq 2}Rj_*j^{-1}(M_S) \rightarrow (M/2M)_{\{0\}}[-2] \xrightarrow{+1}.$$

Note that  $(M/2M)_{\{0\}}[-2] \in {}^{1/2}D_{\mathbb{C}\text{-c}}^{3/2}(A_X)$ . Hence  $M_S \rightarrow {}^{1/2}\tau^{\leq 2}Rj_*j^{-1}(M_S)$  is a monomorphism and an epimorphism in the quasi-abelian category  ${}^{1/2}D_{\mathbb{C}\text{-c}}^2(A_X)$ . Moreover, we have an exact sequence

$$0 \rightarrow M_S \rightarrow {}^{1/2}\tau^{\leq 2}Rj_*j^{-1}(M_S) \rightarrow (M/2M)_{\{0\}}[-2] \rightarrow 0$$

in the abelian category  ${}^{1/2}D_{\mathbb{C}\text{-c}}^{[3/2, 2]}(A_X)$  and an exact sequence

$$0 \rightarrow (M/2M)[-3]_{\{0\}} \rightarrow M_S \rightarrow {}^{1/2}\tau^{\leq 2}Rj_*j^{-1}(M_S) \rightarrow 0$$

in the abelian category  ${}^{1/2}D_{\mathbb{C}\text{-c}}^{[2, 5/2]}(A_X)$ . Note that we have an isomorphism of distinguished triangles

$$\begin{array}{ccccc} \varphi_x(M_S) & \longrightarrow & \varphi_x({}^{1/2}\tau^{\leq 2}Rj_*j^{-1}(M_S)) & \longrightarrow & \varphi_x((M/2M)_{\{0\}}[-2]) \xrightarrow{+1} \\ \downarrow \wr & & \downarrow \wr & & \downarrow \wr \\ M_{\{0\}}[-2] & \xrightarrow{2} & M_{\{0\}}[-2] & \longrightarrow & (M/2M)_{\{0\}}[-2] \xrightarrow{+1} \end{array}$$

Here  $\varphi_x$  is the vanishing cycle functor.

## REFERENCES

- [1] Alexandr A. Beilinson, Joseph Bernstein and Pierre Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), 5–171, Astérisque, **100**, Soc. Math. France, Paris, 1982.
- [2] Tom Bridgeland, *Stability conditions on triangulated categories*, Ann. of Math. (2) **166** (2007), no. 2, 317–345.
- [3] Masaki Kashiwara, *t-structures on the derived categories of holonomic  $\mathcal{D}$ -modules and coherent  $\mathcal{O}$ -modules*, Mosc. Math. J. **4** (2004), no. 4, 847–868.
- [4] ———, *Equivariant derived category and representation of real semisimple Lie groups*, Representation theory and complex analysis, Lecture Notes in Math., **1931**, Springer, Berlin, (2008) 137–234.

- [5] Masaki Kashiwara and Pierre Schapira, *Sheaves on Manifolds*, Grundlehren der Mathematischen Wissenschaften, **292**, Springer-Verlag, Berlin, 1994. x+512 pp.
- [6] Jean-Pierre Schneiders, *Quasi-abelian Categories and Sheaves*, Mém. Soc. Math. Fr. (N.S.) 1999, no. **76**, vi+134 pp.

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